Risk Aversion Pays in the Class of 2 x 2 Games with No Pure Equilibrium

Dirk Engelmann

CERGE-EI

Charles University
Center for Economic Research and Graduate Education
Academy of Sciences of the Czech Republic
Economics Institute

WORKING PAPER SERIES (ISSN 1211-3298)
Electronic Version

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Risk Aversion Pays in the Class of 2×2 Games with No Pure Equilibrium*

Dirk Engelmann†
CERGE-EI
Charles University, Prague, and
the Academy of Sciences of the Czech Republic

June 5, 2003

Abstract

Simulations indicated that, in the class of 2×2 games which only have a mixed equilibrium, payoffs are increased by risk aversion compared to risk neutrality. In this paper I show that the total expected payoff to a player over this class in equilibrium is indeed higher if this player is risk averse than if he is risk neutral provided that all games are played with the same probability. Furthermore, I show that for two subclasses of games more risk aversion is always better, while for a third subclass an intermediate level of risk aversion is preferable.

Simulace ukazují, že v množině her typu 2x2 které mají ekvilibrium pouze ve smíšených strategiích jsou výplaty zvýšeny averzí vůči riziku. V této práci ukazuji, že celková očekávaná výplata hráče v ekvilibriu je v tomto typu her ve skutečnosti vyšší, pokud je tento hráč averzní vůči riziku, než když je vůči riziku neutrální za předpokladu, že jsou všechny hry hrány se stejnou pravděpodobností. Dále ukazuji, že pro dvě podmnožiny těchto her je větší averze vůči riziku vždy lepší, kdežto pro třetí podmnožinu je nejlepší střední úroveň averze.

(JEL Classification: **C72, D89)**
keywords: risk aversion, mixed strategy equilibria

*I thank Steffen Huck, Michael Kosfeld, Wieland Müller, Jörg Oechssler, Sergey Slobodyan, and Martin Strobel for helpful comments. Part of this research was conducted while I was a visitor at the University of Zürich and the California Institute of Technology. I wish to thank these institutions for their hospitality. Financial support by the Deutsche Forschungsgemeinschaft (DFG, grant EN 459/1-1), the DGZ-DekaBank and the CERGE-EI foundation (ESC-postdoctoral fellowship) is gratefully acknowledged.

†CERGE-EI, P.O. Box 882, Politickych veznu 7, 11121 Prague 1, Czech Republic, dirk.engelmann@cerge-ei.cz

1 Introduction

In many areas of economic theory individuals are assumed to be endowed with a utility function expressing their preferences over monetary outcomes. Robson (1992, 1996a, and 1996b) and To (1999) provide evolutionary explanations for the prevalence of specific risk attitudes. Dekel and Scotchmer (1999) study conditions for the selection of risk taking in winner-take-all games. Strobel (2001) shows that for chicken games a payoff monotone dynamic would lead to a population of ever increasing risk taking.

In the study at hand I investigate which kinds of utility functions have the potential to be best suited for promoting long term survival in an evolutionary model based on the class of 2×2 games without pure equilibrium. Specifically I compare the expected monetary payoffs of players with risk averse utility functions to that of players with risk neutral utility functions.

The restriction of the present analysis to the class of 2×2 games without pure equilibrium is motivated by simulations performed by Huck *et al.* (1999). They simulated an evolutionary process selecting between agents with different risk attitudes based on the equilibrium payoffs in randomly generated 2×2 games. These simulations indicated an advantage for risk averse players. The strength of this indication depended on the equilibrium selection criterion applied to games with multiple equilibria. However, the effect of higher long term propagation of risk averse players was particularly pronounced if attention was restricted to the class of 2×2 games with no pure equilibrium. These results yield the intuition that risk averse players receive higher equilibrium payoffs in this class.

In this paper I prove that this intuition is indeed correct. To this aim let all payoffs in a 2×2 game be drawn from a uniform distribution on $[0, 1]$ and consider then only those games with no pure equilibrium. All these games then have the same density. I compare the expected payoff of player 1 for a risk averse type with (fixed) utility function $U(x)$ and a risk neutral type with utility function $V(x) = x$. In a mixed strategy equilibrium the strategy of player 2 is determined by the preferences of player 1 (player 2 chooses probabilities such that player 1 is indifferent), but not by player 2's own preferences. The degree of risk aversion of player 2 is only relevant for player 1's payoff by influencing the mixing probabilities of player 1. This influence of player 2's degree of risk aversion on player 1's payoff will be taken care of by aggregating games to classes of games where the total effect of player 2's degree of risk aversion is neutral.

Let the payoff-matrix of player 1 in a game G where player 1 chooses between T and B and player 2 between L and R be

	L	R
T	a	b
B	c	d

.

Without loss of generality assume $a > \max\{b, c, d\}$. Then $d > b$ holds if there is only a mixed equilibrium. Furthermore, in this case the payoff of player 2 for (T, R) exceeds that for (T, L) and his payoff for (B, L) exceeds that for (B, R).

The relative magnitude of c determines three cases:[1]

1. $a > c > d > b$
2. $a > d > c > b$
3. $a > d > b > c$.

In the following section I show that risk aversion increases the expected payoff compared to risk neutrality in each of the three subclasses of games defined by these cases and thus over the whole class of 2×2 games with no pure equilibrium.

The primary result is in contrast to the effects of risk aversion in bargaining games. As outlined e.g. by Binmore *et al.* (1986), a concave transformation of a player's utility function, i.e. if he becomes more risk averse (or more impatient), changes the Nash bargaining solution in favor of the other player. Increasing a player's risk aversion weakens his bargaining position because the risk of not reaching an agreement becomes more threatening to him. In contrast, in the class of games studied here, the basic intuition (which is entirely correct only in case 1) is that the increased risk aversion of player 1 leads player 2 to increase the probability on player 1's preferred outcome to keep him indifferent, and this benefits player 1. This is, of course, a somewhat perverse argument common for mixed strategy equilibria.

In Section 3, I study the effects of different levels of risk aversion. In particular, I show that in cases 1 and 2 the expected payoff increases with the degree of absolute risk aversion, whereas in case 3 a positive but finite level of risk aversion yields higher expected payoffs than extreme levels of risk aversion. Hence in an evolutionary model based on games of cases 1 and 2 the population would always tend towards consisting of players with more extreme risk aversion, whereas it would

[1] The class of games with ties between payoffs has total mass zero, given the assumption on the distribution of the payoffs. Hence these games can be ignored.

tend towards consisting of players with finite levels of risk aversion if only games of case 3 were played. I also discuss to what extent the assumption of a uniform distribution of payoffs can be relaxed.

Although the class of games dealt with in this paper covers only a minor part of the reality that people face, I believe that this result may give a basis for an evolutionary explanation for the widely observed phenomenon of risk aversion.

Note that nothing is concluded about the utility levels. Hence although a risk averse player would be in advantage compared to a risk neutral player in an evolutionary model where the dynamic is driven by the given payoffs, it cannot be decided whether he also “feels happier”.

2 Analysis

The following proposition contains the main result.

Proposition 1 *The overall effect of risk aversion of a player over the set of* 2×2 *games with no pure equilibrium is an increase of his expected payoff in equilibrium.*

The proof will be conducted separately for each of the three cases defined in the introduction.

Case 1 $a > c > d > b$

Player 1 always prefers that player 2 chooses L since $a > b$ and $c > d$. Let p denote the probability that player 2 chooses L. Hence the expected payoff of player 1 increases with p. Lemma 2 then gives the crucial result.

Lemma 2 *In cases 1 and 2 the equilibrium probability* p *that player 2 chooses* L *is higher if player 1 is risk averse than if player 1 is risk neutral.*

Proof. Since $a > c > b$ and $a > d > b$ the risk associated with the payoffs is higher if player 1 chooses T than if he chooses B. While a risk neutral player 1 will be indifferent between T and B if they yield the same expected payoff, a risk averse player 1 will only be indifferent if T yields a higher expected payoff than B to compensate for the larger associated risk. The latter requires that in equilibrium p is larger if player 1 is risk averse since in case 1 $a - b > c - d$ implies that an increase of p has a larger impact on the expected payoff for T than on that for B, while in case 2 since $a - b > 0 > c - d$ an increase of p will increase the expected payoff for T and decrease that for B. Hence for both cases 1 and 2 player 2 will in equilibrium choose L with a higher probability if player 1 is risk averse than if he is risk neutral. ■

Since in any single game in case 1 the expected payoff to player 1 increases with p, Lemma 2 implies that in equilibrium the expected payoff is higher if player 1 is risk averse than if he is risk neutral.

Case 2 $a > d > c > b$

Let p^* and $\overline{p}$ denote the equilibrium probability that player 2 chooses L if player 1 is risk neutral or risk averse, respectively.

Consider the class of games with fixed payoffs $a > d > c > b$ for player 1 and denote it by $\Gamma := \Gamma_{abcd}$. Note that this is actually a *class* of games since the payoffs for player 2 are not specified. p^* and $\overline{p}$ depend only on a, b, c, d and are thus equal for all games in Γ. By Lemma 2 $\overline{p} > p^*$. Lemma 3 gives the key result for case 2.

Lemma 3 *Over any class Γ the expected payoff to player 1 increases with the probability p that player 2 chooses L, if player 1 chooses his equilibrium strategy.*

Proof. Let q denote the equilibrium probability that player 1 chooses T in a specific game. q depends only on the payoffs to player 2. By exchanging player 2's payoff for (T, L) with that for (B, R) and that for (T, R) with that for (B, L) while leaving the payoffs to player 1 fixed, one obtains another game in class Γ with only a mixed equilibrium and the equilibrium probability of player 1 to choose T is $1 - q$. Since player 2's payoffs are all drawn from the same distribution, both games have the same density. Therefore over the class Γ, q is distributed symmetrically around $\frac{1}{2}$.

The expected payoffs to player 1 for decisions L and R of player 2 in one game are $E(L) = qa + (1 - q)c$ and $E(R) = qb + (1 - q)d$. Therefore the expected (with respect to the distribution of q over a class Γ with density f) payoffs to player 1 over a class of games Γ, given that he plays his equilibrium strategy are

$$\begin{aligned} E_\Gamma(L) &= \int_0^1 (q(a-c)+c)f(q)dq = \int_0^1 cf(q)dq + \int_0^1 q(a-c)f(q)dq \\ &= c + (a-c)\frac{1}{2} = \frac{1}{2}(a+c) \qquad \text{and} \qquad (1) \end{aligned}$$

$$\begin{aligned} E_\Gamma(R) &= \int_0^1 (q(b-d)+d)f(q)dq = \int_0^1 df(q)dq + \int_0^1 q(b-d)f(q)dq \\ &= d + (b-d)\frac{1}{2} = \frac{1}{2}(b+d), \qquad (2) \end{aligned}$$

where (1) and (2) result from the symmetry of the distribution of q around $\frac{1}{2}$ (which implies for the expected value of q, $\int_0^1 qf(q)dq = \frac{1}{2}$). Then

$$E_\Gamma(L) - E_\Gamma(R) = \frac{1}{2}(a+c) - \frac{1}{2}(b+d) = \frac{1}{2}(a-b-(d-c)) > 0 \text{ since } a > d > c > b.$$

Therefore, the expected payoff to player 1 over a class of games Γ, given that he chooses his equilibrium strategy, is higher if player 2 chooses L than if player 2 chooses R and thus increases with p. ■

By Lemma 2 player 2 chooses L with higher probability if player 1 is risk averse than if player 1 is risk neutral, i.e. $\overline{p} > p^*$. Also $\overline{p}$ and p^* are the same for all games in one class Γ. Thus by Lemma 3 for any such class Γ the expected payoff to player 1 in equilibrium is higher if he is risk averse than if he is risk neutral. Since the set of all games in case 2 falls into classes Γ_{abcd}, the expected payoff to player 1 over the whole set is increased by risk aversion.

Case 3 $a > d > b > c$

While the expected payoff for player 1 is increased by risk aversion for any single game in case 1 and over any class of games with fixed payoffs for player 1 in case 2, the situation is less straightforward in case 3.

Consider a game G_1 with payoffs $a_1 > d_1 > b_1 > c_1$ and additionally $a_1 - b_1 > d_1 - c_1$. Then there is a game G_2 with payoffs $a_2 = a_1, b_2 = a_1 - (d_1 - c_1) = c_1 + a_1 - d_1, c_2 = c_1, d_2 = c_1 + a_1 - b_1$. G_2 is also a game of case 3 and due to the fact that all payoffs originally are drawn from the same uniform distribution, G_2 has the same density as G_1 conditioned on the order of payoffs (note that G_2 is obtained from G_1 by shifting d and b within the range given by a and c). From

$a_2 - b_2 = a_1 - (c_1 + a_1 - d_1) = d_1 - c_1$ and $d_2 - c_2 = c_1 + a_1 - b_1 - c_1 = a_1 - b_1$ follows in particular $d_2 - c_2 > a_2 - b_2$.[2]

As in case 2, consider classes Γ_1 and Γ_2 of games of types G_1 and G_2. Correspondingly, let p_i^* and $\overline{p_i}$ denote the equilibrium probability for a choice of L in class Γ_i for the case of risk neutrality and risk aversion, respectively. Then the following three lemmas result.

Lemma 4 *If player 1 is risk neutral, then $p_2^* = p_1^*$. Thus player 2 chooses the same strategy over all games in the classes Γ_1 and Γ_2.*

Proof. In equilibrium player 2 chooses $p_1^* = \frac{d_1-b_1}{d_1-b_1+a_1-c_1} = \frac{1}{1+\frac{a_1-c_1}{d_1-b_1}}$ in all games G_1 in Γ_1 and $p_2^* = \frac{d_2-b_2}{d_2-b_2+a_2-c_2} = \frac{1}{1+\frac{a_2-c_2}{d_2-b_2}}$ in all games G_2 in Γ_2 if player 1 is risk neutral. From $a_2 - c_2 = a_1 - c_1$ and $d_2 - b_2 = c_1 + a_1 - b_1 - (c_1 + a_1 - d_1) = d_1 - b_1$ follows $p_2^* = p_1^*$. ■

Lemma 5 *The total expected payoff to player 1 over both classes Γ_1 and Γ_2 will not be changed if player 2 changes p by the same margin over all games in the union Γ of these classes.*

Proof. The total expected payoffs to player 1 for the choices L and R of player 2 over the classes Γ_1 and Γ_2 are $E_1(L) = \frac{1}{2}(a_1+c_1)$, $E_1(R) = \frac{1}{2}(b_1+d_1)$, $E_2(L) = \frac{1}{2}(a_2+c_2)$ and $E_2(R) = \frac{1}{2}(b_2+d_2)$, respectively (by applying the same argument as in the proof of Lemma 3 to classes Γ_1 and Γ_2 separately). Since the union $\Gamma = \Gamma_1 \cup \Gamma_2$ consists of pairs of games, one in each class, described as above, of the same density, the total expected payoffs for both choices L and R over the union are just $E_\Gamma(L) = E_1(L) + E_2(L)$ and $E_\Gamma(R) = E_1(R) + E_2(R)$. Then

$$\begin{aligned} E_\Gamma(L) - E_\Gamma(R) &= \frac{1}{2}(a_1 + c_1 + a_2 + c_2) - \frac{1}{2}(b_1 + d_1 + b_2 + d_2) \\ &= \frac{1}{2}((a_1 - b_1) - (d_2 - c_2) + (a_2 - b_2) - (d_1 - c_1)) = 0. \end{aligned}$$

Hence to player 1 the total expected payoff over both classes is the same if player 2 chooses always R, always L or chooses some fixed p, as long as it is the same for all games in the union. Consequently, the total expected payoff to player 1 over both classes will not be changed if player 2 changes p by the same margin Δp over all games in the union of classes Γ_1 and Γ_2 because this just implies adding $\Delta p(E_\Gamma(L) - E_\Gamma(R)) = 0$ to the expected payoff. ■

[2] Of course, the situation is completely symmetric in the sense that for any such game G_2 there is a corresponding game G_1 with the given properties. The crucial point is that the whole class of games falls into such pairs.

Lemma 6 *If player 1 is risk averse $\overline{p_1} > \overline{p_2}$, thus player 2 chooses L with higher probability in class Γ_1 than in class Γ_2.*

Proof. If player 1's utility function is given by $U(x)$ then in equilibrium player 2 chooses

$$\overline{p_1} = \frac{U(d_1) - U(b_1)}{U(d_1) - U(b_1) + U(a_1) - U(c_1)} = \frac{1}{1 + \frac{U(a_1) - U(c_1)}{U(d_1) - U(b_1)}}$$

in all games G_1 in Γ_1 and

$$\overline{p_2} = \frac{U(d_2) - U(b_2)}{U(d_2) - U(b_2) + U(a_2) - U(c_2)} = \frac{1}{1 + \frac{U(a_2) - U(c_2)}{U(d_2) - U(b_2)}}$$

in all games G_2 in Γ_2.

Risk aversion implies

$$U(d_2) - U(b_2) < U(d_1) - U(b_1). \tag{3}$$

This is established by the following argument: $d_2 - b_2 = d_1 - b_1$, hence $d_2 + b_1 = d_1 + b_2$. Therefore lottery L_1, which yields both d_2 and b_1 with probability $\frac{1}{2}$, and lottery L_2, which yields both d_1 and b_2 with probability $\frac{1}{2}$, both have the same expected payoff. But since $d_2 > d_1$ and $b_1 < b_2$ the variance of L_1 is higher than that of L_2 and hence a risk averse player will prefer L_2. This in turn implies $\frac{1}{2}U(d_2) + \frac{1}{2}U(b_1) < \frac{1}{2}U(d_1) + \frac{1}{2}U(b_2)$, which yields (3). From (3), $a_2 = a_1$, and $c_2 = c_1$, $\overline{p_1} > \overline{p_2}$ follows immediately. ■

By Lemma 4 $p_1^* = p_2^*$. Thus if player 1 is risk neutral, player 2 chooses L with the same probability in all games in Γ. Lemma 5 implies that the total expected payoff over Γ will not be changed if instead player 2 chooses L with an arbitrary probability as long as it is the same for all games in Γ. In particular this will hold if this probability is $\overline{p_2}$. Now by Lemma 6, if player 1 is risk averse, player 2 chooses L with probability $\overline{p_2}$ for all games in Γ_2 and with probability $\overline{p_1} > \overline{p_2}$ for all games in Γ_1. Over the whole class Γ_1 a choice of L by player 2 is more preferable for player 1 than a choice of R since $E_1(L) - E_1(R) = \frac{1}{2}(a_1 - b_1 - (d_1 - c_1)) > 0$ and hence the expected payoff to player 1 over the class Γ_1 increases with p. Thus the total expected payoff to player 1 over Γ is higher if he is risk averse than if player 2 chooses L with probability $\overline{p_2}$ in all games in Γ, which in turn yields the same total expected payoff as if player 1 is risk neutral. Summarizing, risk aversion raises the total expected payoff to player 1 over a pair of classes Γ_1 and Γ_2 of case 3. Since the set of all games in case 3 falls into such pairs of classes, the overall effect of risk aversion of player 1 in case 3 is an increase of his expected payoff in equilibrium.

The statement of proposition 1 has been shown to be true for all three cases independently and therefore it holds of course for the whole set of 2×2 games without pure equilibrium. Thus the proof is completed.

3 Extensions

There are two obvious directions in which the result of proposition 1 could be generalized. First, one might wonder whether the results also hold for more general distributions than the uniform distribution. The second obvious direction is the comparison between different levels of risk aversion, i.e. the question whether there is an optimal intermediate level of risk aversion or whether the expected payoff is always increasing in the degree of risk aversion.

Consider the second question first; it seems of particular interest if one wants to use the results in evolutionary models for the explanation of specific risk preferences. Assume that U is twice differentiable. The following two propositions then provide answers to this question. They imply that in an evolutionary model (where the unique mixed equilibria are always played and utility functions propagate according to the payoffs of its bearers) based on games of cases 1 and 2 the population would always tend towards consisting of players with more extreme risk aversion, whereas it would tend towards consisting of players with finite levels of risk aversion if only games of case 3 were played. Over the whole class of 2×2 games without pure equilibrium the effect of increasing risk aversion is not clear since the propositions only give rough estimates for the gains and losses due to increasing risk aversion in the different cases, so they do not allow aggregation over all cases. Since the class of games studied is rather special in the first place, however, it is not much more restrictive to consider the subcases separately. The simulation results in Huck *et al.* (1999) suggest that the overall effect of increasing risk aversion is positive.

Proposition 7 *In cases 1 and 2 the expected payoff is increasing in the degree of risk aversion. Hence if only games of these types were played, the most risk averse players would be best off.*

Proof. The proof is a straightforward extension of the proof of proposition 1 for cases 1 and 2. An equivalent for Lemma 2 can be proved by the same argument. In cases 1 and 2 the risk involved in a choice of T is larger than the risk involved in a choice of B. Hence if a player 1 of type A is more risk averse than a player 1 of type B, to be indifferent type A will require a higher compensation, i.e. a higher expected payoff for T than type B. Now as in the proof of Lemma 2, this implies that p will be larger if player 1 is of type A (i.e. more risk averse) than if he is of type B. That the expected payoff is increasing in p for every single game in case 1 is unrelated to the degree of risk aversion and hence it is settled that the expected payoff in any single game of case 1 increases with the degree of risk aversion. Lemma 3 does not depend on the degree of risk aversion either and thus over all games in case 2 an increase in p increases the expected payoff to player 1. Hence player 1's expected payoff over the whole class of games of case 2 increases in the degree of risk aversion. ■

Proposition 8 *In case 3, if player 1's Arrow-Pratt coefficient of absolute risk aversion* $r_A(x) = -\frac{U''(x)}{U'(x)}$ *tends towards infinity, his expected payoff gain compared to risk neutrality tends towards* 0*, whereas it is larger than some positive value* γ *for at least some intermediate values of* r_A*. This implies that in case 3 the expected payoffs for some positive but finite levels of risk aversion exceed those for extreme levels of risk aversion by a positive value.*

Proof. The proof is based on Lemmas 10 and 11 below. Lemma 10 shows that the expected payoff gain that risk aversion yields over risk neutrality can be made arbitrarily small (i.e. smaller than any $\varepsilon > 0$) by choosing $r_A(x)$ sufficiently large. On the other hand Lemma 11 shows that for absolute risk aversion $\frac{1}{3} < r_A(x) < \frac{5}{3}$ for all x, the gain in expected payoffs is larger than some $\gamma > 0$. Thus by choosing $\varepsilon = \frac{\gamma}{2}$ the expected payoff in the latter case exceeds that in case of a degree of risk aversion as implied by Lemma 10 by more than $\frac{\gamma}{2} > 0$. Note, however, that even excessive degrees of risk aversion still yield a higher expected payoff than risk neutrality. ■

In order to prove Lemma 10, the following lemma is needed.

Lemma 9 *For all* $\varepsilon > 0$ *and* $\eta > 0$ *there exists* $K(\varepsilon, \eta)$ *such that if* $r_A(x) > K(\varepsilon, \eta)$ *for all* x*, then for all games of case 3 with* $b - c > \eta$*, the probability that player 2 chooses* L *will be* $p < \frac{\varepsilon}{2}$*.*

Proof. Let $r_A(x) = -\frac{U''(x)}{U'(x)} > K = K(\varepsilon, \eta)$ for all x. This yields

$$U''(x) < -KU'(x). \tag{4}$$

Let $e = \frac{b+c}{2}$. With (4) and $U'' < 0$ then follows

$$\begin{aligned} U'(b) &= U'(e) + \int_e^b U''(x)dx < U'(e) - K \int_e^b U'(x)dx < U'(e) - K(b-e)U'(b) \\ &\Rightarrow (1 + K(b-e))\, U'(b) < U'(e). \end{aligned}$$

The probability that player 2 chooses L is given by

$$\begin{aligned} p &= \frac{U(d) - U(b)}{U(d) - U(b) + U(a) - U(c)} = \frac{\int_b^d U'(x)dx}{\int_b^d U'(x)dx + \int_c^a U'(x)dx} \\ &< \frac{(d-b)U'(b)}{\int_b^d U'(x)dx + \int_c^a U'(x)dx} < \frac{(d-b)U'(b)}{\int_c^a U'(x)dx} < \frac{(d-b)U'(b)}{\int_c^e U'(x)dx} \\ &< \frac{(d-b)U'(b)}{(e-c)U'(e)} < \frac{(d-b)}{(e-c)} \frac{1}{K(b-e)} = \frac{4(d-b)}{K(b-c)^2} < \frac{4}{K\eta^2} \\ &= \frac{\varepsilon}{2} \text{ for } K = K(\varepsilon, \eta) = \frac{8}{\varepsilon \eta^2}. \quad \blacksquare \end{aligned}$$

Lemma 10 *For $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if player 1's coefficient of absolute risk aversion $r_A(x) > K(\varepsilon)$ for all x, then his expected payoff gain compared to risk neutrality over all games of case 3 is $< \varepsilon$.*

Proof. Consider games of classes Γ_1 and Γ_2 as defined in the proof for case 3 in section 2. First choose $\eta(\varepsilon)$ sufficiently small such that the games with $b_1 - c_1 < \eta(\varepsilon)$ have a mass of at most $\frac{\varepsilon}{2}$. For games of classes Γ_1 and Γ_2 obviously $\overline{p_1} - \overline{p_2} \leq 1$ and the maximal gain for player 1 if player 2 chooses L instead of R is also 1. Hence (by application of Lemmas 4 and 5) the maximal total gain in expected payoffs of a risk averse compared to a risk neutral player 1 over these games (with $b_1 - c_1 < \eta(\varepsilon)$) is $\frac{\varepsilon}{2}$.

Now consider games with $b_2 - c_2 > b_1 - c_1 > \eta(\varepsilon) > 0$. By Lemma 9, for all these games if $r_A(x) > K(\varepsilon, \eta(\varepsilon))$ for all x, then p will be smaller than $\frac{\varepsilon}{2}$. Lemma 6 shows for any specific utility function that $\overline{p_1} > \overline{p_2}$. This yields for any utility function with $r_A(x) > K(\varepsilon, \eta(\varepsilon))$ that $0 < \overline{p_1} - \overline{p_2} < \frac{\varepsilon}{2}$. The payoff gain from a choice of L by player 2 compared to R is at most 1 and hence the total gain over classes Γ_1, Γ_2 of games with $b_2 - c_2 > b_1 - c_1 > \eta(\varepsilon)$ in expected payoffs is $< \frac{\varepsilon}{2}$. Hence if $r_A(x) > K(\varepsilon) = K(\varepsilon, \eta(\varepsilon))$, then the total gain over all games of case 3 compared to risk neutrality is $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\blacksquare$

Lemma 11 *If for player 1's coefficient of absolute risk aversion* $\frac{1}{3} < r_A(x) < \frac{5}{3}$ *for all* x*, then his total expected payoff over all games of case 3 exceeds that for risk neutrality by at least* γ *with* $\gamma > 2^{-28}$.

Proof. The values $\frac{1}{3}$ and $\frac{5}{3}$ are somewhat arbitrary and are chosen for convenience. They only serve to show that for some intermediate level of risk aversion the gains are above some positive lower bound.

Lemma 6 implies that for any pair of classes of subgames there is a gain from risk aversion over risk neutrality. Hence to find a lower bound for the total gains from a specific degree of risk aversion one can limit the attention to some of these classes in order to simplify the proof. (Note that this leaves out a large part of the gains. In addition, most of the inequalities below are very rough. Hence the actual gains over all games in case 3 will be much larger than γ.)

Consider pairs of classes of games with $a > \frac{7}{8}, \frac{1}{4} < d_1 < \frac{3}{8}, c < \frac{1}{16} < b_1 < \frac{1}{8}$. This implies (with $b_2 = c + a - d_1$) $b_2 - d_1 > \frac{1}{8}$ and $d_1 - b_1 > \frac{1}{8}$. With $\frac{1}{3} < r_A(x) < \frac{5}{3} \Leftrightarrow -\frac{5}{3}U'(x) < U''(x) < -\frac{1}{3}U'(x)$ one obtains the following two auxiliary inequalities

$$\begin{aligned}
U'(b_2) &= U'(d_1) + \int_{d_1}^{b_2} U''(x)dx < U'(d_1) - \frac{1}{3}\int_{d_1}^{b_2} U'(x)dx \\
&< U'(d_1) - \frac{1}{3}(b_2 - d_1)U'(b_2) \\
&\Leftrightarrow U'(b_2) < \frac{U'(d_1)}{1 + \frac{1}{3}(b_2 - d_1)} \\
&\Rightarrow U'(d_1) - U'(b_2) > \frac{\frac{1}{3}(b_2 - d_1)}{1 + \frac{1}{3}(b_2 - d_1)} U'(d_1) \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
U'(d_1) &= U'(c) + \int_{c}^{d_1} U''(x)dx > U'(c) - \frac{5}{3}\int_{c}^{d_1} U'(x)dx \\
&> U'(c) - \frac{5}{3}(d_1 - c)U'(c) = \left(1 - \frac{5}{3}(d_1 - c)\right) U'(c). \qquad (6)
\end{aligned}$$

Now the difference between the probabilities for L in classes Γ_1 and Γ_2 is

$$
\begin{aligned}
\overline{p_1} - \overline{p_2} &= \frac{U(d_1) - U(b_1)}{U(d_1) - U(b_1) + U(a) - U(c)} - \frac{U(d_2) - U(b_2)}{U(d_2) - U(b_2) + U(a) - U(c)} \\
&= \frac{(U(a) - U(c))\,(U(d_1) - U(b_1) - U(d_2) + U(b_2))}{(U(d_1) - U(b_1) + U(a) - U(c))\,(U(d_2) - U(b_2) + U(a) - U(c))} \\
&> \frac{(U(a) - U(c))\,(U(d_1) - U(b_1) - U(d_2) + U(b_2))}{4\,(U(a) - U(c))^2} \\
&= \frac{U(d_1) - U(b_1) - U(d_2) + U(b_2)}{4\,(U(a) - U(c))} = \frac{\int_{b_1}^{d_1} U'(x)dx - \int_{b_2}^{d_2} U'(x)dx}{4\int_c^a U'(x)dx} \\
&> \frac{(d_1 - b_1)(U'(d_1) - U'(b_2))}{4\int_c^a U'(x)dx} = \frac{1}{4}\frac{(d_1 - b_1)(U'(d_1) - U'(b_2))}{\int_c^{d_1} U'(x)dx + \int_{d_1}^a U'(x)dx} \\
&> \frac{1}{4}\frac{(d_1 - b_1)(U'(d_1) - U'(b_2))}{(d_1 - c)U'(c) + (a - d_1)U'(d_1)} \overset{(5)}{>} \frac{1}{4}\frac{(d_1 - b_1)\frac{\frac{1}{3}(b_2 - d_1)}{1 + \frac{1}{3}(b_2 - d_1)}U'(d_1)}{(d_1 - c)U'(c) + (a - d_1)U'(d_1)} \\
&\overset{(6)}{>} \frac{1}{4}\frac{(d_1 - b_1)\frac{\frac{1}{3}(b_2 - d_1)}{1 + \frac{1}{3}(b_2 - d_1)}U'(d_1)}{\frac{(d_1 - c)}{1 - \frac{5}{3}(d_1 - c)}U'(d_1) + (a - d_1)U'(d_1)} \\
&> \frac{1}{4}\frac{\frac{1}{8}\frac{1}{25}}{2} = 5^{-2}2^{-6}.
\end{aligned}
$$

If player 2 chooses L instead of R in class Γ_1, player 1's expected payoff increases by

$$
\frac{1}{2}\left(a - b_1 - (d_1 - c)\right) > \frac{1}{2}\left(\frac{3}{4} - \frac{3}{8}\right) = \frac{3}{16}.
$$

By Lemmas 4 and 5 the expected payoff would be equal to that in the case of risk neutrality if player 2 chose L with probability $\overline{p_2}$ in both classes Γ_1 and Γ_2. Thus since player 2 chooses $\overline{p_1}$ in class Γ_1, player 1's expected payoff over any pair of classes Γ_1 and Γ_2 exceeds that of risk neutrality by at least

$$
(\overline{p_1} - \overline{p_2})\frac{3}{16} > 5^{-2}2^{-6}\frac{3}{16} = 5^{-2}2^{-10}3.
$$

The total mass of the classes of games considered is at least $\left(\frac{1}{8}\right)^2\left(\frac{1}{16}\right)^2 = 2^{-14}$. Hence the total gain in expected payoffs over all games of case 3 by absolute risk aversion of $\frac{1}{3} < r_A(x) < \frac{5}{3}$ for all x compared to risk neutrality is (much) larger than $\gamma := 5^{-2}2^{-24}3 > 2^{-28}$. ■

Concerning the question whether the results also hold if the payoffs are drawn from more general distributions than the uniform distribution, it is straightforward that in cases 1 and 2 the results do not depend on the distribution as long as all payoffs are drawn from the same distribution. Since Lemma 2 holds for any single game it is completely independent of the distribution.

Lemma 3 only uses the assumption that the payoffs are all drawn from the same distribution without making any requirements on its form. The results do not, however, extend to case 3. This is because in case 3 there are classes of games where risk aversion increases the expected payoff as well as those where it decreases expected payoffs. The way to get around this problem in the proof of proposition 1 for case 3 is to pair up classes which yield a total positive effect of risk aversion. This, however, requires that in each pair of games both games have the same density. For this to hold for all classes requires a uniform distribution. Alternative ways to aggregate games do not yield the desired results either. For example if the distribution is symmetric around $\frac{1}{2}$ one can pair a game in Γ_1 with another game that is obtained by reflecting the payoffs around $\frac{1}{2}$, i.e. $a_2 = 1 - c_1, b_2 = 1 - d_1, c_2 = 1 - a_1, d_2 = 1 - b_1$. The equivalence of Lemmas 4 and 5 then hold, but that of Lemma 6 does not.

Summing up these two generalizations yields that if only games of cases 1 and 2 are considered, then more risk averse players are always in advantage to less risk averse players, independent of the distribution that the payoffs are drawn from. If, however, games of case 3 are considered, then while there is an overall positive effect of risk aversion compared to risk neutrality in the case of a uniform distribution, this result does not easily extend to more general distributions and more risk aversion leads to lower expected payoffs if it exceeds some finite level.

There are further possible extensions of the present analysis. Extending the strategy space is likely to yield qualitatively similar results, but will require the consideration of many more subcases, including differentiating between mixed and completely mixed equilibria. The same holds for multi-player games. Given the weak results of the simulations in Huck *et al.* (1999) eliminating the restriction to games without pure equilibria will, even if the result still holds, probably render the proof much less straightforward. This is in particular the case since in the class of games studied by Strobel (2001) risk taking is favored.

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CERGE-EI
P.O.BOX 882
Politických vezòù 7
111 21 Prague 1
Czech Republic
http://www.cerge-ei.cz