A Temporary Equilibrium Model of Asset Pricing[1]

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Abstract

This paper examines a decision-making problem of rational agents with risk averse utilities in the financial market both in statics and in dynamics. In the financial market there are two securities, one risky security and one riskless bond, and a continuum of investors with heterogeneous preferences, endowments, and beliefs. Given that investors' beliefs are described by Gamma distribution with different parameters, predictions are made about competitive equilibrium asset prices and the sizes of groups of traders and of their positions in the market. The dynamic extension of the model shows how traders' beliefs can be updated by Bayes' rule. This rational updating determines the next period investors' beliefs and predicts future equilibrium asset prices and the positions of traders in the market.

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1. Introduction

Modeling decision-making problems under uncertainty enriches the theoretic treatment of reality and brings us closer to the ideal model of social-economic arrangements in society. In particular, the functioning of modern financial markets can be understood and improved upon only by incorporating elements of randomness, heterogeneity, and asymmetry of information into the agents' decision problem. The model in this paper deals with these aspects of uncertainty to predict both investors' positions in the market and asset prices.

In general, the model is based on the following settings. A continuum of traders who have different preferences, different endowments, and different information about other investors characteristics' trade in the financial market. Differences in information cause investors to form different expectations about future asset prices. The assumption of the agents' heterogeneity has not been fully theoretically explored and at the same time has always been empirically tractable (see for example, Smith, Suchanek, Williams [22]).

Each investor's financial decision on optimal portfolio holdings is made under uncertainty about tomorrow's security prices. If an investor purchases a risky asset or a riskless bond, both of which are traded in the market, then an other investor sells them. Both investors make decisions in a rational manner. Perhaps, the buyer/seller thinks that there is a high/low probability that the prices of shares will go up. Thus, one distinguishes three main reasons for trading: different expectations, different risk aversion, and different endowments. With initial endowments given and initial beliefs exogenously predetermined, asset prices in the two markets and investors' income are simultaneously endogenously determined. This model explains the short-run behavior of asset prices when asset returns and consequently investors' income come only from capital gains and not from dividends. Traders in the markets make decisions by looking in advance only to the next period, and each investor is interested only in the value of his own portfolio.

In contrast, Lucas [21] shows that the stream of future discounted real dividends determines the asset price. The key to his result is that he considers an infinitely lived representative agent who is concerned only about real dividends and that the state processes are known to the representative agent. However, in reality the assumption of homogeneity is very restrictive; and the short-run gains/losses from real dividends are incomparably lower than the short-run possible gains/losses from changes in asset prices. Not surprisingly, Lucas' model did not find much empirical support. (See, for example, Mehra and Prescott [18]).

My work is related to that of Green [16], Grandmont [13], and Werner [24] in the sense that individual beliefs differ across investors. In my model, individual beliefs have the same support while this assumption is not necessary in the models of Green, Grandmont, and Werner. Given the assumption that agents in my model have overlapping expectations, I can concentrate on individual beliefs as the major determinant of equilibrium asset prices in the short-run.

The paper is organized as follows. The first section considers a parametric economic model which specifies particular functional forms for investors' utility and their probability beliefs about the next period price. I derive the optimal decision rule for a rational agent who chooses optimal holdings of risky and riskless securities. It is assumed that investors have a constant absolute risk aversion utility function and traders' probability beliefs about next period prices is described by a Gamma distribution with different parameters for each agent.

In the second section, I make additional assumptions about the distributions of individual probability beliefs and initial portfolio holdings across investors and predict the next period portfolio holdings across investors and equilibrium asset price.

In the third section, I extend the static model to a dynamic case. Using the rules for the Bayesian updating of probability beliefs, it is possible to determine the parameters of the distribution of probability beliefs at any moment in time after estimating the parameters of the initial probability beliefs and observing the whole price history. I specify the Bayesian updating mechanism for individual beliefs and derive an asset pricing formula for each discrete period of time. One of the predictions in this section is that there will be a positive autocorrelation in the asset pricing process.

Finally, I make concluding remarks and identify directions for future research.

2. The Economic Model

Let us consider a financial market with an infinitely lived riskless bond and risky security. A continuum of risk averse investors trade in the market and make decisions about optimal portfolio holdings looking only one period ahead. Each investor is endowed with an initial portfolio and has expectations about the next period's risky asset price. The investors' initial expectations about the next period price and the initial holdings of bonds and risky assets differ across investors and are exogenous. The competitive equilibrium model in the financial market considered below shows that beliefs about the next period price affect today's competitive equilibrium price.

The investors are assumed to have non-homogeneous beliefs about the next period prices because of differences in their information. The agents' beliefs are represented by a probability distribution over the next period price. There are two ways to specify the distribution of investors' beliefs. One is to assume that each agent's belief is given non-parametrically and to introduce a prior/posteriori distribution on the space of probability measures which describes the distribution of before - trade/after - trade beliefs in the market. This approach is very general, and it requires Bayesian updating in a non-parametrically specified functional space. (See Dubins [8] and Ferguson [10] for more details). The second way is to assume that each agent's probability distribution belongs to a given parametric family of distributions and each agent's belief is described by the parameters of this distribution. Then we have to construct the prior and posteriori distributions over the parameter space which describe the before and after - trade distribution of investors' beliefs in the economy. In what follows, the second way has been adopted for simplicity.

Suppose the measure space $(I, \aleph, \mu)$ is the investors' space, and μ is an atomless probability measure on the investors' space. Each investor in the space is described by three different characteristics. Each agent $i \in I$ has a constant absolute risk averse utility function:

$$u_i(w) = 1 - e^{-a_i w},$$

where $a_i > 0$ represents a coefficient of absolute risk aversion, and w is the next period wealth, which is the market price of the next period portfolio. Without any loss of generality, the price of the riskless bond is normalized to unity. The trader's objective is to maximize the expected utility of the next period wealth.

Each investor's beliefs about the next period risky asset, p, is represented by a Gamma distribution with parameters α and β_i, where α is the shape parameter and is common for all investors and β_i is the scale parameter which describes the agent's beliefs and differs across investors. Given that an investor's belief is described by a gamma distribution, the expected next period price $E(p)$, and expected volatility $V(p)$ are given as follows: $E(p) = \frac{\alpha}{\beta_i}$, and $V(p) = \frac{\alpha}{\beta_i^2}$. Many empirical studies (see, for example, A. Lo and A. MacKinlay [19],[20]) support the idea that the distribution of asset prices has heavy tales. Therefore, it is appropriate to choose the Gamma distribution for modelling asset prices. [2] The investor's budget set can be specified as follows:

$$B_i\left(p^*, \overline{x}_i, \overline{b}_i\right) = \left\{(x_i,\ b_i) \in R_+^2 \mid p^* x_i + b_i = p^*\ \overline{x}_i + \overline{b}_i\ \right\},$$

where $p^* > 0$ denotes today's actual price of the risky asset; x_i and b_i denote the feasible amounts of the risky asset and the riskless bond, respectively, for investor i; $\overline{x}_i$ denotes investor i's initial holdings of the risky asset; $\overline{b}_i$ denotes investor i's initial holdings of the riskless bond; and $p^*\ \overline{x}_i + \overline{b}_i$ is equal to the investor's initial wealth, $\overline{w}_i$.

Given these assumptions, the space of investors' characteristics is a four-dimensional Euclidean space with the following elements: $\left(a_i, \beta_i, \overline{x}_i, \overline{b}_i\right)$, for $i \in I$.

Investor i's maximization problem can be written as:

$$\max \int_0^{\infty} \left(1 - e^{-a_i(px_i+b_i)}\right)\ g\left(p \mid \alpha,\, \beta_i\right)$$

$$s.t. \qquad x_i,\ b_i\ \in\ B_i\left(p^*, \overline{x}_i, \overline{b}_i\right),$$

where $g\left(p \mid \alpha,\, \beta_i\right)$ is the Gamma density function.

The maximization yields investor i's optimal demand for the risky and riskless securities given in equations (1.1)-(1.3). If the individual characteristics are such that the coefficient describing the agent's beliefs, β_i, belongs to the interval $\left(\frac{\alpha}{p^*}; +\infty\right)$, then the agent's demand is equal to

$$x_i = 0, \qquad b_i = p^*\ \overline{x}_i + \overline{b}_i \qquad \text{if} \qquad \frac{\alpha}{p^*} < \beta_i. \tag{1.1}$$

Equation (1.1) shows that the total agent's wealth is invested only in the riskless bond. The condition

$$\frac{\alpha}{p^*} < \beta_i \Leftrightarrow E\left(p\right) = \frac{\alpha}{\beta_i} < p^*$$

states that if the investor's expected next period price of the risky asset is less than today's price, p^*, the investor prefers not to buy the risky asset and to invest all his current wealth in the riskless security, no matter what the coefficient of absolute risk aversion is. All investors whose beliefs belong to the interval

[2] See Hogg, Craig [17] for more details about the properties of this distribution.

$\left(\frac{\alpha}{p^*};+\infty\right)$ become net suppliers of the risky asset and sell all of the risky assets that they hold to invest their total wealth in the riskless bond.

If the investors' characteristics are such that the coefficient describing agent beliefs, β_i, belongs to the interval $\left(\frac{\alpha}{p^*} - a\ \overline{\overline{x}}_i - \frac{a\ \overline{\overline{b}}_i}{p^*}; \frac{\alpha}{p*}\right]$, then the agent's demand is equal to

$$x_i = \frac{\alpha}{a_i p^*} - \frac{\beta_i}{a_i}\ , \quad b_i = p^*\ \overline{\overline{x}}_i + \overline{\overline{b}}_i - \frac{\alpha}{a_i} + p^* \frac{\beta_i}{a_i} \tag{1.2}$$

$$\text{if } \frac{\alpha}{p^*} - a_i\ \overline{\overline{x}}_i - \frac{a_i\ \overline{\overline{b}}_i}{p^*} < \beta_i \leq \frac{\alpha}{p*}.$$

Equation (1.2) shows that the total wealth is invested in both securities. The second condition can be divided into two parts:

(A) $\frac{\alpha}{p*} - a_i\ \overline{\overline{x}}_i < \beta_i \leq \frac{\alpha}{p*}$ and (B) $\frac{\alpha}{p*} - a_i\ \overline{\overline{x}}_i - \frac{a_i\ \overline{\overline{b}}_i}{p*} < \beta_i \leq \frac{\alpha}{p*} - a_i\ \overline{\overline{x}}_i$.

If inequality (A) is satisfied, then the left-hand side of (A) implies that

$$x_i - \overline{\overline{x}}_i = \frac{\alpha}{a_i p*} - \frac{\beta_i}{a_i} - \overline{\overline{x}}_i = \frac{1}{a_i}\left(\frac{\alpha}{p*} - a_i\ \overline{\overline{x}}_i - \beta_i\right) < 0.$$

The term in brackets is negative because $\frac{\alpha}{p^*} - a_i\ \overline{\overline{x}}_i < \beta_i$. At the same time, the left-hand side of condition (A) implies that

$$b_i - \overline{\overline{b}}_i = p^*\ \overline{\overline{x}}_i - \frac{\alpha}{a_i} + p^* \frac{\beta_i}{a_i} = \frac{p^*}{a_i}\left(-\frac{\alpha}{p*} + a_i\ \overline{\overline{x}}_i + \beta_i\right) > 0.$$

These, together with $\beta_i \leq \frac{\alpha}{p*}$, means that investors with the characteristics given by condition (A) hold the total wealth in both securities, and they are the net suppliers of the risky asset and the net demanders of riskless bond in the market.

In the case when inequality (B) is satisfied, then the following two inequalities hold:

$$\begin{aligned} x_i - \overline{\overline{x}}_i &= \frac{\alpha}{a_i p^*} - \frac{\beta_i}{a_i} - \overline{\overline{x}}_i = \frac{1}{a_i}\left(\frac{\alpha}{p^*} - a_i\ \overline{\overline{x}}_i - \beta_i\right) \geq 0, \\ b_i - \overline{\overline{b}}_i &= p^*\ \overline{\overline{x}}_i - \frac{\alpha}{a_i} + p^* \frac{\beta_i}{a_i} = \frac{p^*}{a_i}\left(-\frac{\alpha}{p^*} + a_i\ \overline{\overline{x}}_i + \beta_i\right) \leq 0. \end{aligned}$$

This means that the investors with characteristics satisfying the right-hand side of condition (B) is a demander of the risky asset and a supplier of the risky bond. The left-hand side of condition (B) means that investors whose characteristics satisfy the inequality $\beta_i > \frac{\alpha}{p^*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p^*}$ diversify their wealth between both securities available in the market.

If the investors' characteristics are such that the coefficient describing the agent's beliefs, β_i, belongs to the interval $\left(0, \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*}\right]$, then the agent's demand for a risky asset and riskless bond is equal to

$$x_i = \overline{\overline{x}}_i + \frac{\overline{\overline{b}}_i}{p*}, \quad b_i = 0 \quad \text{if} \quad \beta_i \leq \frac{\alpha}{p*} - a_i\ \overline{\overline{x}}_i - \frac{a_i\ \overline{\overline{b}}_i}{p*} \tag{1.3}$$

Equation (1.3) shows that the total wealth is invested in the risky asset. The condition $\beta_i \leq \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*}$, derived from the investor's optimization problem, implies that those investors whose characteristics satisfy this condition are the suppliers of all the bond holdings and the demanders of the risky asset. These investors invest all of their wealth totally in the risky asset. (For a full derivation see Appendix 1.)

The case when $\overline{\overline{x}}_i = \overline{\overline{b}}_i = 0$ is ruled out in the model. (If $\overline{\overline{x}}_i = \overline{\overline{b}}_i = 0$, the agent can not participate in trading because of a short sales restriction.)

The above result shows that the agent's characteristics $\left(a_i, \beta_i, \overline{\overline{x}}_i, \overline{\overline{b}}_i\right)$, which differ across investors, determine the optimal demand and supply of risky and riskless securities. Four cases are possible:

1) In the first case, investor i expects that tomorrow's risky asset price will go down. As a result, he prefers to sell his holdings of the risky asset and keep all his wealth in the riskless bond.

2) In the second case, investor i sells a part of his risky asset holdings and invests his wealth in the riskless bond.

3) In the third case, investor i sells a part of his bond holdings and invests his wealth in the risky securities.

4) In the forth case, investor i sells his bond holdings and invests his total wealth in the risky security.

3. Comparative Statics

This section considers the comparative statics of the model, i.e., it shows how the exogenous variables of the model - α, a, m, n, r, s, l, k - affect endogenous variables such as:

1) the size of investor groups whose portfolio contain either only riskless bonds, or only risky assets or both risky and riskless securities;

2) the size of investor groups who are suppliers of risky assets or suppliers of bonds; and

3) the equilibrium price for risky asset.

Assume that a probability measure on the space of investor characteristics $\left(a_i, \beta_i, \overline{x}_i, \overline{b}_i\right) \in R_+^4$ - is described as follows:

1) β_i , investor i's belief about the next period price, has the Gamma distribution with parameters m and n, i.e., $\beta_i \sim \Gamma(m,n)$;

2) $\overline{x}_i$, the initial holdings of the risky asset by investor i , has the Gamma distribution with parameters r and s, i.e., $\overline{x}_i \sim \Gamma(r,s)$;

3) $\overline{b}_i$, the initial holdings of the riskless bond by investor i , has the Gamma distribution with parameters l and k, i.e., $\overline{b}_i \sim \Gamma(l,k)$;

4) a_i , the coefficient of investor's absolute risk aversion, is constant across investors and $a_i = a$ for all $i \in I$.

The ratio of the distribution parameters $\frac{m}{n}$ indicates the average belief about the quantity $\frac{Ep}{Vp}$; (where $\frac{Ep}{Vp}$ represents the ratio of the next period average price to its volatility) and $\frac{m}{n^2}$ indicates the dispersion of beliefs across investors. The ratios of the distribution parameters $\frac{r}{s}$, and $\frac{l}{k}$ describe the average holdings of the risky asset and riskless bond across investors.

Proposition 2.1

The size of the group of investors whose wealth is allocated in the riskless bond only in the end of the period decreases (increases) if either of α or k increases (either m or a increases) and does not depend on r and l.

Proof:

In this proof, the property of an incomplete gamma function is used to rewrite the formula determining the size of the group of investors who invest their wealth only in the riskless bond as follows[3]:

[3] The size of the group of investors who in the end of the period have all their wealth invested

$$P_1 = \Pr\left(\beta_i > \frac{\alpha}{p^*}\right) = 1 - \Pr\left(\frac{p^*\beta_i}{a} \leq \frac{\alpha}{a}\right) = 1 - \frac{k^m}{\Gamma(m)} \int_0^{\frac{\alpha}{a}} t^{m-1} e^{-kt} dt = 1 - \gamma\left(m, k, \frac{\alpha}{a}\right). \tag{2.1}$$

Recall from the investor's optimization problem that the condition $\left(\beta_i > \frac{\alpha}{p^*}\right)$ means that $p^* > E_i(p)$. Those investors who fall in the category with such expectations about the next period price sell all of their risky asset holdings to invest all of their wealth in the riskless bond. Note that this happens irrespective of their other characteristics including the coefficient of the absolute risk aversion, a, initial asset holdings $\overline{x}_i$, and bond holdings $\overline{b}_i$. As defined above, the probability includes neither the average risky asset holdings, x, nor the average bond holdings, b. Hence, the size of this group is not affected by exogenously determined average risky asset holdings or by the average riskless bond holdings.

The term $\frac{k^m}{\Gamma(m)} \int_0^{\frac{\alpha}{a}} t^{m-1} e^{-kt} dt$ is the incomplete gamma function, $\gamma\left(m, k, \frac{\alpha}{a}\right)$, with parameters $m, k, \frac{\alpha}{a}$. From the properties of the incomplete gamma function it is easy to verify that the size of this group, $\Pr\left(\beta_i > \frac{\alpha}{p*}\right)$, decreases if either α or k goes up; the size of group $\Pr\left(\beta_i > \frac{\alpha}{p*}\right)$ increases if either m or a increases. (The properties of the incomplete gamma function are given in more detail in Appendix 2.) Q.E.D.

Proposition 2.2

The size of the group of investors who hold both types of securities at the end of the period increases if k, m, r, or l increases.

Proof:

This proof follows directly from the validity of the two previous propositions. Recall from the investor's optimization problem that the size of the group of investors, whose end of period portfolio holdings contain both the riskless bond and the risky asset, is determined by the size of the investors' characteristics satisfying

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in the riskless bond is determined by the size of the investors' characteristics given by the inequality: $\beta_i > \frac{\alpha}{p^*}$.

the following condition: $\left(\frac{\alpha}{a} < \frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i\right)$. From this it follows that the size of this group of investors depends on the probability measure defined on the space of investors' characteristics.

$$P_2 = \Pr\left(\frac{\alpha}{a} < \frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i\right) = \tag{2.2}$$

$$= \Pr\left(\frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i\right) - \Pr\left(\frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right) =$$

$$= \Pr\left(\frac{p^*\beta_i}{a} \leq \frac{\alpha}{a}\right) - \Pr\left(\frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right),$$

where $\Pr\left(\frac{p^*\beta_i}{a} \leq \frac{\alpha}{a}\right)$ indicates the size of the group of investors whose end of period portfolio contains either only risky assets or both risky and riskless securities. The expression $\Pr\left(\frac{p*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right)$ is the size of the group of investors whose end of period portfolio contains only risky assets. The above expression can be rewritten in terms of the probabilities determined in Propositions 1 and 2:

$$\Pr\left(\frac{p^*\beta_i}{a} \leq \frac{\alpha}{a}\right) - \Pr\left(\frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right) =$$

$$= 1 - \left[\Pr\left(\frac{p^*\beta_i}{a} > \frac{\alpha}{a}\right) + \Pr\left(\frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right)\right]. \tag{2.3}$$

Equation (2.3) can be evaluated from the results of Propositions 1 and 3. The term in brackets decreases if k, m, r, or l increases; consequently, the value of expression (2.3) increases if k, m, r, or l increases. Q.E.D.

Proposition 2.3

The size of the individual groups who choose only risky securities in trading increases if α increases, and decreases if m, r, l, n, or a increases.

Proof:

From the previous section (where the optimal holdings were determined), it follows that if the condition

$$\beta_i \leq \frac{\alpha}{p^*} - a\,\overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p^*}$$

is satisfied about a trader's characteristics then a given trader will sell of all bonds and hold only risky securities.

is satisfied. This condition can be rewritten as

$$\frac{p^*\beta_i}{a} + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{}$$

α goes up, while it decreases if m, r, l, k or a goes up. Details of the derivation of this proposition are demonstrated in Appendix 2. Q.E.D.

Proposition 2.4

The size of the group of investors who are the net suppliers of the risky assets in the market increases (decreases) if r, m, k, or a increases (α increases).

Proof:

An investor sells all risky assets if his belief satisfies the inequality $\beta_i > \frac{\alpha}{p*}$. This inequality means that the expected next period price of a security, $\frac{\alpha}{\beta_i}$, is less than today's price p^*. The investor is the seller of a part of his own risky assets if his characteristics satisfy the inequality $\frac{\alpha}{p*} - a\ \overline{\overline{x}}_i < \beta_i \leq \frac{\alpha}{p*}$. From these two expressions, it follows that the investor is the net supplier of the risky assets if the following inequality is satisfied: $\frac{p*\beta_i}{a} + p^*\ \overline{\overline{x}}_i > \frac{\alpha}{a}$. Given that β_i and $\overline{\overline{x}}_i$ are random variables with a Gamma distribution, then the sum $\frac{p*\beta_i}{a} + p^*\ \overline{\overline{x}}_i$ is a random variable which has a Gamma distribution with parameters $m + r$ and k. Therefore, the size of the group of investors who are net suppliers of the risky asset is

$$\Pr\left\{\frac{p^*\beta_i}{a} + p^*\ \overline{\overline{x}}_i > \frac{\alpha}{a}\right\} = \frac{k^{m+r}}{\Gamma\left(m+r\right)} \int\limits_{\frac{\alpha}{a}}^{\infty} t^{m+r-1}e^{-kt}. \tag{2.5}$$

From the properties of the Gamma function (described in detail in Appendix 2), it follows that the probability defined in expression (2.5) increases/decreases if m, r, k, or a increases/α increases. Thus, Proposition 4 holds. Q.E.D.

Proposition 2.5

The size of the group of investors who are net suppliers of the riskless bond decreases (increases) if r, m, k, or a increases (α increases).

Proof:

An investor sells a part of his riskless bonds and demands the risky securities if his characteristics satisfy the inequality $\frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*} < \beta_i \leq \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i$. A more convenient way to rewrite this inequality is $\frac{\alpha}{p*} < \beta_i + a\ \overline{\overline{x}}_i + \frac{a\,\overline{\overline{b}}_i}{p*} \leq \frac{\alpha}{p*} + \frac{a\,\overline{\overline{b}}_i}{p*}$. On the other hand, the investor sells all his initial holdings of bonds if his characteristics satisfy the inequality $\beta_i + a\ \overline{\overline{x}}_i + \frac{a\,\overline{\overline{b}}_i}{p*} \leq \frac{\alpha}{p*}$.

Combining the last two inequalities, it is obvious that the size of the group of investors who are excess suppliers of the riskless bond is determined by the probability measure of the set of investor characteristics which satisfy the inequality

$$\beta_i + a\ \overline{\overline{x}}_i + \frac{a\ \overline{\overline{b}}_i}{p^*} \leq \frac{\alpha}{p^*} + \frac{a\ \overline{\overline{b}}_i}{p^*}. \tag{2.6}$$

Using the fact that $\beta_i + a\ \overline{\overline{x}}_i$ has a Gamma distribution with parameters $m+x$ and k, the size of the group of investors who are net suppliers of the riskless bond is equal to

$$\Pr\left\{\frac{p^*\beta_i}{a} + p^*\ \overline{\overline{x}}_i \leq \frac{\alpha}{a}\right\} = \frac{k^{m+r}}{\Gamma(m+r)} \int_0^{\frac{\alpha}{a}} t^{m+r-1} e^{-kt}. \tag{2.7}$$

Clearly, from this expression it follows that this probability decreases (increases) if either m, or r, or k, or a increases (α increases). Q.E.D.

Proposition 2.6

The competitive equilibrium price of the risky asset

1) increases if the expectation about the next period increases (α increases);

2) decreases if the average β_i across investors increases (which means that $\frac{m}{n}$ increases);

3) decreases if the average initial asset holdings increase (which means that $\frac{r}{s}$ increases);

4) increases if the average initial bond holdings increase (which means that $\frac{l}{k}$ increases);

5) decreases if the coefficient of the absolute risk aversion, a, increases.

Proof:

Recall from Section 1 of this paper that the optimal demands for the risky asset and the riskless bond for investor i are given by

1) $x_i = 0 \qquad b_i = p^*\ \overline{\overline{x}}_i + \overline{\overline{b}}_i \qquad$ if $\qquad \frac{\alpha}{p*} < \beta_i$;

2) $x_i = \frac{\alpha}{ap*} - \frac{\beta_i}{a} \qquad b_i = p^*\ \overline{\overline{x}}_i + \overline{\overline{b}}_i - \frac{\alpha}{a} + p^* \frac{\beta_i}{a} \qquad$ if $\qquad \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*} < \beta_i \leq \frac{\alpha}{p*}$;

3) $x_i = \overline{\overline{x}}_i + \frac{\overline{\overline{b}}_i}{p*} \qquad b_i = 0 \qquad \beta_i \leq \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*}$.

From the assumption about the distribution of investor characteristics it follows that average belief and the average initial asset and bond holdings are given respectively by

$$\int_I \beta_i d\mu(i) = \tfrac{m}{n},$$
$$\int_I \overline{\overline{x}}_i\ d\mu(i) = \tfrac{r}{s},$$
$$\int_I \overline{\overline{b}}_i\ d\mu(i) = \tfrac{l}{k},$$

From this, it follows

$$\int_I x_i d\mu(i) = 0 \times \Pr\left\{\frac{\alpha}{p*} < \beta_i\right\} + \left(\frac{\alpha}{ap*} - \frac{m}{na}\right) \times \tag{2.8}$$

$$\times \Pr\left\{\frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\ \overline{\overline{b}}_i}{p*} < \beta_i \leq \frac{\alpha}{p*}\right\} + \left(\frac{r}{s} + \frac{l}{kp^*}\right)\Pr\left\{\beta_i \leq \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\ \overline{\overline{b}}_i}{p*}\right\}.$$

After aggregating the individuals' demand for the risky asset, it follows:

$$\int_I x_i d\mu(i) = \left(\frac{\alpha}{ap*} - \frac{m}{na}\right) \times P_2 + \left(\frac{r}{s} + \frac{l}{kp^*}\right) \times P_3.$$

The last equation gives the mean demand for the risky asset. The mean excess demand for the risky asset is

$$\left(\frac{\alpha}{ap*} - \frac{m}{na}\right) \times P_2 + \left(\frac{r}{s} + \frac{l}{kp^*}\right) \times P_3 - \int_I \overline{\overline{x}}_i\ d\mu(i) =$$

$$= \left(\frac{\alpha}{ap*} - \frac{m}{na}\right) \times P_2 + \left(\frac{r}{s} + \frac{l}{kp^*}\right) \times P_3 - \frac{r}{s}. \tag{2.9}$$

To determine the competitive equilibrium asset price, the mean excess demands for the risky security should be set equal to zero:

$$\left(\frac{\alpha}{ap*} - \frac{m}{na}\right) \times P_2 + \left(\frac{r}{s} + \frac{l}{kp^*}\right) \times P_3 - \frac{r}{s} = 0$$

The solution to these equations – the competitive equilibrium price, p^*– is given by

$$p^* = \frac{\frac{\alpha}{a}P_2 + \frac{l}{k}P_3}{\frac{r}{s}(1 - P_3) + \frac{m}{an}P_2} = \frac{\frac{\alpha}{a}P_2 + \frac{l}{k}P_3}{\frac{r}{s}(P_1 + P_2) + \frac{m}{an}P_2}. \tag{2.10}$$

It is straightforward to verify from this expression that Proposition 6 is true. Q.E.D.

4. The Dynamic Model

The dynamic model is an extension of the static model described above. Since we have considered infinitely long - lived securities in the financial market, a short - run dynamic model is a finite sequence of overlapping two - period models and the dynamics of the security price is a sequence of temporary equilibrium prices. In this section we also consider changes in investor beliefs over time due to information acquisition. The main issue of the dynamic model is the effect of the realization of the competitive equilibrium price on investors' beliefs. In particular after observing current security price traders update their priory beliefs and the high realization of current price leads that investors in the market have stochastically high beliefs. In the dynamic version of the model, the distribution of posterior beliefs about the next period price across investors has the same functional form as the distribution of prior beliefs in the market.

Proposition 3.1

If investors' prior beliefs, β_i, have a Gamma distribution, then their posterior beliefs will again have a Gamma distribution with different parameters.

Proof:

In the dynamic setting, the after-trade distribution of investor beliefs β_i after observing the realization of the security price of the risky asset is

$$g\left(\beta_i \mid p\right) = \frac{Ga\left(p \mid \alpha, \beta_i\right) Ga\left(\beta_i \mid m, n\right)}{\int\limits_0^\infty Ga\left(p \mid \alpha, \beta_i\right) Ga\left(\beta_i \mid m, n\right) d\beta_i}. \tag{3.1}$$

Using the property of a gamma distribution one can obtain

$$\int\limits_0^\infty Ga\left(p \mid \alpha, \beta_i\right) Ga\left(\beta_i \mid m, n\right) d\beta_i = \int\limits_0^\infty \frac{\beta_i^\alpha}{\Gamma\left(\alpha\right)} p^{\alpha-1} e^{-\beta_i p} \frac{n^m}{\Gamma\left(m\right)} \beta_i^{m-1} e^{-\beta_i n} d\beta_i = \tag{3.2}$$

$$= \frac{p^{\alpha-1}}{\Gamma\left(\alpha\right)} \frac{n^m}{\Gamma\left(m\right)} \int\limits_0^\infty \beta_i^\alpha e^{-\beta_i p} \beta_i^{m-1} e^{-\beta_i n} d\beta_i = \frac{p^{\alpha-1}}{\Gamma\left(\alpha\right)} \frac{n^m}{\Gamma\left(m\right)} \int\limits_0^\infty \beta_i^{\alpha+m-1} e^{-\beta_i (p+n)} d\beta_i =$$

$$= \frac{p^{\alpha-1}}{\Gamma\left(\alpha\right)} \frac{n^m}{\Gamma\left(m\right)} \frac{\Gamma\left(\alpha+m\right)}{\left(p+n\right)^{\alpha+m}}.$$

If we substitute Expression (3.2) in the Expression (3.1), one can get,

$$g(\beta_i \mid p) = \frac{Ga(p \mid \alpha, \beta_i)\, Ga(\beta_i \mid m, n)}{\int\limits_0^\infty Ga(p \mid \alpha, \beta_i)\, Ga(\beta_i \mid m, n)\, d\beta_i} =$$

$$= \frac{\frac{\beta_i^\alpha}{\Gamma(\alpha)} p^{\alpha-1} e^{-\beta_i p} \frac{n^m}{\Gamma(m)} \beta_i^{m-1} e^{-\beta_i n}}{\frac{p^{\alpha-1}}{\Gamma(\alpha)} \frac{n^m}{\Gamma(m)} \frac{\Gamma(\alpha+m)}{(n+p)^{(m+\alpha)}}} = \frac{(n+p)^{m+\alpha}}{\Gamma(m+\alpha)} \beta_i^{m+\alpha-1} e^{-\beta_i (p+n)}. \quad (3.3)$$

this expression (3.3) indicates that the after-trade beliefs also have a Gamma distribution with parameters $m+\alpha$ and $n+p$.

Now it becomes clear that the model can be used to make predictions about the next period's asset price. In particular, if parameters $m_t, n_t, r_t, s_t, l_t, k_t, P_{1t}, P_{2t}, P_{3t}$ and α are estimated according to the procedure described in Appendix 3 of this paper, one can use them together with the competitive equilibrium price observed in the market to predict the next period's new parameters of the distribution of investor beliefs.

$$\hat{m}_{t+1} = \hat{m}_t + \hat{\alpha} \text{ and } \hat{n}_{t+1} = \hat{n}_t + p_t \quad (3.4)$$

for any $t = 1, 2...T$. Consequently one can predict the next period's competitive equilibrium price. From this it follows that today's price has an effect on tomorrow's expectations about the next period price. When today's competitive equilibrium risky asset price, p_t, is high, then n_{t+1} increases. This leads to a stochastic shift in investors' beliefs. Using formulas (2.10) and (3.4), one can easily verify that the price p_{t+1} increases. This means that today's high price leads to tomorrow's stochastically high beliefs, and consequently, a high price. This prediction is consistent with the empirical research, which supports the finding that there exists a positive correlation between asset prices. In particular, the empirical studies of Lo and MacKinlay [19], [20], and DeBondt and Thaler [6], [7], show that in the short - run (i.e., daily or weekly) asset prices are positively autocorrelated. (In contrast, empirical evidence has been found, for example by Fama and French [9], indicating that asset prices are significantly negatively serially correlated in the long - run.)

In the dynamic model, the competitive equilibrium asset price is a sequence of the static solutions. For any time period, t, the solution is

$$p_t^* = \frac{\frac{\alpha}{a} P_{2t} + \frac{l_t}{k_t} P_{3t}}{\frac{r_t}{s_t}(P_{1t} + P_{2t}) + \frac{m_t}{a n_t} P_{2t}}. \quad (3.5)$$

This formula assumes that parameters α and a are constants in the short-run, and n_t and m_t are determined from (3.4) as follows: $n_t = n_{t-1} + p_{t-1}$ and $m_t = m_{t-1} + \alpha$. Equation (3.5), determining the dynamic competitive equilibrium asset price, shows that the current price of the risky asset is explained by its previous price; a high price at period $(t-1)$ implies a high value of the current price. There is a positive serial autocorrelation in the price process.

5. Concluding Remarks and Directions for the Future Research

The main goals of this paper were to consider a parametrized economy in order to study and derive optimal solutions for the portfolio holding problem. In this paper exogenous variables such as beliefs and initial portfolio holdings are shown to affect equilibrium price.

The particular functional specifications of investor utilities, beliefs, along with the specification of the distribution functions on the space of investor characteristics make it possible to solve the model and derive the closed - form solution for the equilibrium price. Apart from this, testable predictions about the effects of the exogenous parameters of the model (the coefficient of absolute risk aversion, the average belief, the average asset holding, and the average bond holding) on the sizes of the groups of investors who have different portfolio holdings and on the equilibrium asset price were made.

In the dynamic extension of the model, individual beliefs change over time; therefore, the high realization of current price implies stochastically high investor beliefs. In this setting, it is enough to know the initial distribution of beliefs and the current competitive equilibrium asset price to make predictions about the next period competitive equilibrium asset price. The dynamic extension of the model reveals the existence of a positive autocorrelation pattern in the price process.

Possible directions for future theoretical research would be to consider the model in continuous time to combine into the model factors explaining the movements of asset prices in the short-run with the fundamentals lying behind the long-run behavior of asset prices and to measure temporary deviations of stock prices from their fundamental values.

6. Appendices

Appendix 1.

Trader i's maximization problem can be written as:

$$\max \int_0^\infty \left(1 - e^{-a_i(px_i + b_i)}\right) g\left(p \mid \alpha,\, \beta_i\right) dp$$
$$s.t. x_i, b_i \in B_i\left(p^*, \overline{\overline{x_i}}, \overline{\overline{b_i}}\right),$$

where

$$g\left(p \mid \alpha,\, \beta_i\right) = \frac{\beta_i^\alpha}{\Gamma(\alpha)} p^{\alpha-1} e^{-\beta_i r}.$$

The first order condition for optimization is

$$\int_0^\infty e^{-a_i\left(px_i - p^* x_i + p^* \overline{\overline{x}}_i + \overline{\overline{b}}_i\right)} a_i \left(p - p^*\right) \frac{\beta_i^\alpha}{\Gamma(\alpha)} p^{\alpha-1} e^{-\beta_i p} dp = 0.$$

This equation can be written as

$$\int_0^\infty e^{-a_i p x_i} \left(p - p^*\right) p^{\alpha-1} e^{-\beta_i p} dp = 0,$$

$$\int_0^\infty e^{-a_i p x_i} p\, p^{\alpha-1} e^{-\beta_i p} dp = p^* \int_0^\infty e^{-a_i p x_i} p^{\alpha-1} e^{-\beta_i p} dp = 0.$$

Integrating separately the left and right integrals, it follows

$$I_1 = \int_0^\infty e^{-a_i p x_i} p\, p^{\alpha-1} e^{-\beta_i p} dp = \int_0^\infty e^{-p(a_i x_i + \beta_i)} p^\alpha dp.$$

After changing the variable $p\left(a_i x_i + \beta_i\right) = y$, one obtains

$$I_1 = \int_0^\infty \frac{y^\alpha}{\left(a_i x_i + \beta_i\right)^\alpha}\, e^{-y} \frac{dy}{\left(a_i x_i + \beta_i\right)} = \frac{1}{\left(a_i x_i + \beta_i\right)^{\alpha+1}} \int_0^\infty y^\alpha\, e^{-y} dy =$$

$$= \frac{\Gamma(\alpha+1)}{(a_i x_i + \beta_i)^{\alpha+1}}.$$

Analogously integrating the right part of the equation, one obtains

$$I_2 = p^* \int_0^\infty e^{-a_i p x_i} p^{\alpha-1} e^{-\beta_i p} dp = p^* \int_0^\infty e^{-p(a_i x_i + \beta_i)} p^{\alpha-1} dp.$$

After changing the variable $p(a_i x_i + \beta_i) = y$, one obtains

$$I_2 = p^* \int_0^\infty e^{-p(a_i x_i + \beta_i)} p^{\alpha-1} dp = p^* \int_0^\infty \frac{y^{\alpha-1}}{(a_i x_i + \beta_i)^{\alpha-1}} e^{-y} \frac{dy}{(a_i x_i + \beta_i)} =$$

$$= p^* \frac{1}{(a_i x_i + \beta_i)^{\alpha}} \int_0^\infty y^{\alpha-1} e^{-y} dy = p^* \frac{\Gamma(\alpha)}{(a_i x_i + \beta_i)^{\alpha}}.$$

The first order condition becomes

$$\frac{\Gamma(\alpha+1)}{(a_i x_i + \beta_i)^{\alpha+1}} = p^* \frac{\Gamma(\alpha)}{(a_i x_i + \beta_i)^{\alpha}}.$$

Using the property of a Gamma function $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, one obtains

$$\frac{\alpha}{a_i x_i + \beta_i} = p^*.$$

From the last equation, it follows that the optimal demand for the risky asset and for the riskless bond for investor i is

1) $x_i = 0, \quad b_i = p^* \bar{x}_i + \bar{b}_i \quad$ if $\quad \frac{\alpha}{p*} < \beta_i$;

2) $x_i = \frac{\alpha}{a_i p*} - \frac{\beta_i}{a_i}$, $b_i = p^* \bar{x}_i + \bar{b}_i - \frac{\alpha}{a_i} + p^* \frac{\beta_i}{a_i} \quad$ if $\quad \frac{\alpha}{p*} - a\, \bar{x}_i - \frac{a\, \bar{b}_i}{p*} < \beta_i \leq \frac{\alpha}{p*}$;

3) $x_i = \bar{x}_i + \frac{\bar{b}_i}{p*}$, $\quad b_i = 0 \quad \beta_i \leq \frac{\alpha}{p*} - a\, \bar{x}_i - \frac{a\, \bar{b}_i}{p*}$.

Appendix 2.

The incomplete gamma functions can be written as

$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \quad$ for $\quad (a, x > 0)$, and

$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt \quad$ for $\quad (a, x \geq 0)$.

Clearly $\gamma(a,x) = \Gamma(a) - \Gamma(a,x)$.

For more details about the incomplete Gamma function, see Gautschi [8], [9].

Let us denote $G(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1}e^{-t}dt$ and $G^*(a,x) = \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1}e^{-t}dt$.

If the function e^{-t} is expanded, it can be easily verified that

$$G(a,x) = \frac{x^a}{\Gamma(a)}\left(\frac{1}{a} - \frac{x}{1!(a+1)} + \frac{x^2}{2!(a+2)} + ...\right);$$

this series converges rapidly if x is small. Integration by part leads to

$$G^*(a,x) = \frac{e^{-x}}{\Gamma(a)}\left(x^{a-1} + (a-1)x^{a-2} + (a-1)^2 x^{a-3} + ...\right).$$

From this equation, it follows that

$$G^*(a,x) = G^*(a-1,x) + \frac{e^{-x}x^{a-1}}{\Gamma(a)}.$$

Using the last expression, it is easy to see that $G^*(a,x)$ is an increasing function with respect to a. In the model, $G^*(m,\alpha) = \Pr\left(\beta_i > \frac{\alpha}{p*}\right)$ and is an increasing function of m. The second relationship can be easily verified by an variable transformation:

$$\Pr\left(\beta_i > \frac{\alpha}{p*}\right) = \frac{k^m}{\Gamma(m)}\int_{\frac{\alpha}{p*}}^{\infty} t^{m-1}e^{-kt}dt = \frac{1}{\Gamma(m)}\int_{\frac{\alpha}{p*}}^{\infty}(tk)^{m-1}e^{-kt}d(kt) = \frac{1}{\Gamma(m)}\int_{\alpha k}^{\infty} y^{m-1}e^{-y}dy.$$

The value of the last integral clearly decreases if k increases. Q.E.D.

Appendix 3.

This appendix summarizes (see the table below) the effects that the exogenous variables in the model $(\alpha, a, m, n, r, s, l, k)$ have on the endogenous variables (the sizes of the groups of investors with different holdings of market securities, and on the equilibrium risky asset price). The signs indicated in the table show the direction of these effects. These comparative statics constitute the main testable predictions of the model considered in the first two sections of this paper. To test these predictions it is necessary to know the values of the exogenous and endogenous variables. While the endogenous variables are observable, the values of the exogenous parameters have to be statistically estimated by means of the likelihood function technique. This appendix proposes econometric techniques to estimate parameter values of the model. It is straightforward to estimate values of r, s, l, k after observing initial portfolio holdings. However, the estimation of parameters α, a, m, and n, of the distribution of investor initial beliefs β_i requires specific econometric techniques because β_i is not directly observable.

Table 1[7]

	Parameters							
Endogenous Variables	α	a	m	n	r	s	l	k
1)The size of the individual groups who hold only risky assets	+	—	—	—	—	—	—	—
2)The size of the individual groups who hold only riskless bonds	—	+	+	—	*	—	*	—
3)The size of the individual groups who hold both risky and riskless securities	*	*	*	+	*	+	*	+
4)The size of the individual groups who are excess suppliers of the risky asset	—	+	+	+	+	+	*	+
5)The size of the individual groups who are excess suppliers of the riskless bond	+	—	—	—	—	—	*	—
6)The price of the risky security	+	—	—	+	—	+	+	—

Firstly, consider a method of estimating the exogenous parameters r, s, l, k by

[7] The sign + or - indicates the direction of changes of the endogeneous variable if the exogenous variable increases. The sign * indicates that there is no relationship between exogenous and endogenous variables.

observing the initial portfolio holdings. According to assumptions (2) and (3) stated in Section 2, $\bar{x}_i$, and $\bar{b}_i$ have a Gamma distribution

$$\bar{x}_i \sim \Gamma(r, s), \bar{b}_i \sim \Gamma(l, k).$$

This property of the initial portfolio holdings is used to construct the Maximum Likelihood Estimators (MLE) for the shape parameter, r, and the scale parameter, s. The procedure for estimating parameters l, and k is analogous.

Below the estimates that will maximize the likelihood function of a random sample are determined. Using the density function of a Gamma distribution for a random sample $(\bar{x}_1, \bar{x}_2, ... \bar{x}_n)$, the likelihood function can be written as

$$L(\bar{x}_1, ... \bar{x}_n | r, s) = \prod_{i=1}^{n} g(\bar{x}_i | r, s) = \tag{A.3.1}$$

$$= \left(\frac{s^r}{\Gamma(r)}\right)^n (\bar{x}_1 \bar{x}_2 ..... \bar{x}_n)^{r-1} e^{-s \sum_{i=1}^{n} \bar{x}_i}.$$

After taking the logarithmic transformation, one obtains

$$\ln L = n[r \ln s - \ln \Gamma(r)] + (r-1)\ln(\bar{x}_1 \bar{x}_2 ..... \bar{x}_n) - s\sum_{i=1}^{n} \bar{x}_i = \tag{A.3.2}$$

$$= n\left[r \ln s - \ln \Gamma(r) + (r-1)\ln^n \sqrt{\bar{x}_1 \bar{x}_2 ..... \bar{x}_n} - s\frac{1}{n}\sum_{i=1}^{n} \bar{x}_i\right].$$

The first order conditions for the optimal parameters, r and s, are

$$\frac{\partial \ln L}{\partial x} = n\left[\ln s - \frac{d \ln \Gamma(r)}{dx} + \ln^n \sqrt{\bar{x}_1 \bar{x}_2 ..... \bar{x}_n}\right], \tag{A.3.3}$$

$$\frac{\partial \ln L}{\partial s} = n\left[\frac{r}{s} - \frac{1}{n}\sum_{i=1}^{n} \bar{x}_i\right]. \tag{A.3.4}$$

From the second equation, the ML estimators of the parameters x and s should satisfy the equation

$$\frac{\overset{\curlywedge}{r}}{\overset{\curlywedge}{s}} = \frac{1}{n}\sum_{i=1}^{n} \overline{\overline{x}}_i \ . \tag{A.3.5}$$

The logaritmic transformation of equation (A.3.5) leads to

$$\ln \overset{\curlywedge}{s} = \ln \overset{\curlywedge}{r} - \ln A, \tag{A.3.6}$$

where $A = \frac{1}{n} \sum_{i=1}^{n} \overline{\overline{x}}_i$ denotes the arithmetic mean of the observed asset holdings across investors. Using first order conditions (A.3.3) (A.3.4), the ML estimator for the parameter r satisfies

$$\ln \overset{\curlywedge}{r} - \ln A - \frac{d \ln \Gamma(r)}{dr} + \ln^n \sqrt{\overline{\overline{x}}_1 \overline{\overline{x}}_2 \ ..... \ \overline{\overline{x}}_n} = 0 \tag{A.3.7}$$

This equation can be simplified by denoting the geometric mean of the initial asset holdings, $\sqrt[n]{\overline{\overline{x}}_1 \overline{\overline{x}}_2 \ ..... \ \overline{\overline{x}}_n}$, by G and rearranging terms:

$$\ln \overset{\curlywedge}{r} - \frac{d \ln \Gamma\left(\overset{\curlywedge}{r}\right)}{dr} = \ln \frac{A}{G}. \tag{A.3.8}$$

As a result, the ML estimators of the parameters x and s are functions of the sample arithmetic mean and the sample geometric mean, and they satisfy the following equations:

$$\ln \overset{\curlywedge}{r} - \frac{d \ln \Gamma\left(\overset{\curlywedge}{r}\right)}{dr} = \ln \frac{A}{G} \quad \text{and} \quad \frac{\overset{\curlywedge}{r}}{\overset{\curlywedge}{s}} = \frac{1}{n}\sum_{i=1}^{n} \overline{\overline{x}}_i \ . \tag{A.3.9}$$

For more details about the MLE procedure for the Gamma distribution parameters see, Hogg and Craig [14]. The above described ML estimation of the parameters x and s is directly applicable for estimating the parameters l and k.

Next turn to the estimation of the parameters m and n, α and a. If the quantity β_i, which describes the agent's belief about the next period price, were observable, then the above specified MLE procedure would have been applicable. However, β_i is not directly observable. That is why a new estimation procedure is developed below.

Firstly, recall from the solution to the agent's optimization problem in Section 1 that the optimal individual demand for a risky asset x_i is given by

(1) $x_i = 0$ if $\frac{\alpha}{p*} < \beta_i$;

(2) $x_i = \frac{\alpha}{ap*} - \frac{\beta_i}{a}$ if $\frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*} < \beta_i \leq \frac{\alpha}{p*}$;

(3) $x_i = \overline{\overline{x}}_i + \frac{\overline{\overline{b}}_i}{p*}$ if $\beta_i \leq \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*}$.

By observing that an investor does not hold the risky asset, i.e., $x_i = 0$, one can conclude that $\beta_i \in \left(\frac{\alpha}{p*}, \infty\right)$. By observing that the investor does not hold the riskless bond, i.e., $x_i = \overline{\overline{x}}_i + \frac{\overline{\overline{b}}_i}{p*}$, one can conclude that $\beta_i \in \left(0, \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*}\right]$. In the last case, when the investor holds a positive amount of both securities, one can conclude that the opposite market value of the wealth invested in the risky asset, $-px_i$, has a three parameter Gamma distribution across investors:

$$-px_i \sim \Gamma\left(m, \frac{np*}{a}, \frac{\alpha}{a}\right).[8]$$

The quantity $\frac{p*\beta_i}{a} - \frac{\alpha}{a}$ is observable only in the case when the investor holds both securities, i.e., in case (2). In the other two cases, when $\beta_i \in \left(0, \frac{\alpha}{p*} - a\ \overline{\overline{x}}_i - \frac{a\,\overline{\overline{b}}_i}{p*}\right] \cup \left(\frac{\alpha}{p*}, \infty\right)$, the quantity $\frac{p*\beta_i}{a} - \frac{\alpha}{a}$ is not observable. This means that the problem of censored observations from the left and right sides is present.

Suppose that the after - trade portfolio holdings are known. Let

- n_1 denote the number of investors who have portfolios containing only risky securities,

- n_2 denote the number of investors who have portfolios containing only riskless securities, and

- n_3 denote the number of investors who hold both types of securities. In addition, suppose that for those investors who hold both types of securities the market value of their wealth invested in the risky asset, $p^* x_i$, is known.

As shown in the last footnote, the quantity $\frac{p*\beta_i}{a} - \frac{\alpha}{a}$ has a gamma distribution with the shift parameter $-\frac{\alpha}{a}$. Then the likelihood function can be written as

$$L = \left[\prod_{i=1}^{n_3} g\left(z_i | m, n, -\frac{\alpha}{a}\right)\right] \left[\Pr\left(\frac{p^*\beta_i}{a} > \frac{\alpha}{a}\right)\right]^{n_2} \left[\Pr\left(\frac{p^*\beta_i}{a} + p^*\ \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right)\right]^{n_1}, \tag{A.3.10}$$

where $z_i = -\,p^* x_i$. Using the fact that z_i has three parameter Gamma distribution, and β_i, $\overline{\overline{x}}_i$, and $\overline{\overline{b}}_i$ have the Gamma distribution, the following holds:

[8] This follows from the fact that $-p^* x_i = -\frac{\alpha}{a} + \frac{p*\beta_i}{a}$. By assumption (1) in Section 1 of this chapter, $\beta_i \sim \Gamma(m, n)$. From this it follows that $\frac{p*\beta_i}{a} \sim \Gamma\left(m, \frac{p*n}{a}\right)$. Hence, $-p^* x_i$ has the Gamma distribution with three parameters: m, $\frac{p*n}{a}$ and the shift parameter $\frac{\alpha}{a}$.

$$\prod_{i=1}^{n_3} g\left(z_i|m,n,-\frac{\alpha}{a}\right) = \left(\frac{m^n}{\Gamma\left(m\right)}\right)^{n_3} \prod_{i=1}^{n_3}\left(z_i+\frac{\alpha}{a}\right)^{m-1} e^{-\sum_{i=1}^{n_3}\left(z_i+\frac{\alpha}{a}\right)n}$$

$$\Pr\left(\frac{p^{*}\beta_i}{a} > \frac{\alpha}{a}\right) = \frac{n^m}{\Gamma\left(m\right)} \int\limits_{\frac{\alpha}{a}}^{\infty} t^{m-1} e^{-nt} dt$$

and

$$\Pr\left(\frac{p^{*}\beta_i}{a} + p^{*}\ \overline{\overline{x}}_i + \overline{\overline{b}}_i \leq \frac{\alpha}{a}\right) = \frac{n^{m+x+b}}{\Gamma\left(m+x+b\right)} \int\limits_{0}^{\frac{\alpha}{a}} t^{m+x+b-1} e^{-nt} dt$$

Using the last three expressions in the likelihood function and taking the logaritmic transformation of them, one obtains:

$$\ln L = n_3\left(n\ln m - \ln\Gamma\left(m\right)\right) + \left(m-1\right)\ln\left(\prod_{i=1}^{n_3}\left(z_i+\frac{\alpha}{a}\right)\right) - n\sum_{i=1}^{n_3}\left(z_i+\frac{\alpha}{a}\right) - \tag{A.3.11}$$

$$-n_2\ln\Gamma\left(m\right) + n_2\ln\Gamma\left(m, n\frac{\alpha}{a}\right) - n_1\ln\Gamma\left(m+x+b\right) + n_1\ln\gamma\left(m+x+b, n\frac{\alpha}{a}\right),$$

where the values of x and b are obtained from the estimation procedure described above; $\Gamma\left(m,\frac{n\alpha}{a}\right) = \int\limits_{n\frac{\alpha}{a}}^{\infty} t^{m-1}e^{-t}dt$, and $\gamma\left(m,\frac{n\alpha}{a}\right) = \int\limits_{0}^{\frac{\alpha n}{a}} t^{m-1}e^{-t}dt$. Hence, $\Gamma\left(m,\frac{n\alpha}{a}\right) + \gamma\left(m,\frac{n\alpha}{a}\right) = \Gamma\left(m\right)$.

Differentiating the log-likelihood function with respect to parameters m, n and $\frac{\alpha}{a}$, the first order conditions are obtained:

$$\frac{\partial \ln L}{\partial n} = \ln m - \frac{1}{n_3}\sum_{i=1}^{n_3}\left(z_i+\frac{\alpha}{a}\right) + \frac{n_2}{n_3}\frac{d\ln\Gamma\left(m,n\frac{\alpha}{a}\right)}{dn} + \frac{n_1}{n_3}\frac{d\ln\gamma\left(m+x+b,n\frac{\alpha}{a}\right)}{dn}; \tag{A.3.12}$$

$$\frac{\partial \ln L}{\partial m} = \frac{n}{m} - \frac{d \ln \Gamma(m)}{dm} + \ln^{n_3} \sqrt{\left(\prod_{i=1}^{n_3} \left(z_i + \frac{\alpha}{a} \right) \right)} - \frac{n_2}{n_3} \frac{d \ln \Gamma(m)}{dm} + \quad \text{(A.3.13)}$$

$$+ \frac{n_2}{n_3} \frac{d \ln \Gamma\left(m, n\frac{\alpha}{a}\right)}{dm} - \frac{n_1}{n_3} \frac{d \ln \Gamma(m+x+b)}{dm} + \frac{n_1}{n_3} \frac{d \ln \gamma\left(m+x+b, n\frac{\alpha}{a}\right)}{dm};$$

$$\frac{\partial \ln L}{\partial \frac{\alpha}{a}} = \frac{m-1}{n_3} \left(\sum_{i=1}^{n_3} \frac{1}{\left(z_i + \frac{\alpha}{a}\right)} \right) - n + \frac{n_2}{n_3} \frac{d \ln \Gamma\left(m, n\frac{\alpha}{a}\right)}{d\frac{\alpha}{a}} + \frac{n_1}{n_3} \frac{d \ln \gamma\left(m+x+b, n\frac{\alpha}{a}\right)}{d\frac{\alpha}{a}}. \quad \text{(A.3.14)}$$

Clearly, it is impossible to find a closed - form mathematical solution to the above system of equations. An algoritmic solution has to be considered.

Algorithmic Choice

Let the following quantities be defined:

$$A\left(z, \frac{\alpha}{a}, n_3\right) = \frac{1}{n_3} \sum_{i=1}^{n_3} \left(z_i + \frac{\alpha}{a} \right) \quad \text{modified arithmetic mean,}$$

$$G\left(z, \frac{\alpha}{a}, n_3\right) = \sqrt[n_3]{\left(\prod_{i=1}^{n_3} \left(z_i + \frac{\alpha}{a} \right) \right)} \quad \text{modifed geometric mean,}$$

$$H\left(z, \frac{\alpha}{a}, n_3\right) = \frac{1}{n_3} \left(\sum_{i=1}^{n_3} \frac{1}{\left(z_i + \frac{\alpha}{a}\right)} \right) \quad \text{modifed harmonic mean,}$$

where the three means are functions of $\frac{\alpha}{a}$, and fortunately, $A\left(z, \frac{\alpha}{a}, n_3\right)$ has $\frac{\alpha}{a}$ as an isolated term. A three step numerical solution is considered below.

Step 1:

The first step of the algorithm is based on the fact that $p^* x_i < \frac{\alpha}{a}$ for each observation. That is why the starting value for $\frac{\alpha}{a}$ is taken to be the maximum of $p^* x_i$, i.e., $\left(\hat{\frac{\alpha}{a}}\right)_0 = \max_i \left(p^* x_i\right)$. For the given value of $\left(\hat{\frac{\alpha}{a}}\right)_0$, calculate values for $A\left(z, \frac{\alpha}{a}, n_3\right)$, $G\left(z, \frac{\alpha}{a}, n_3\right)$, and for $H\left(z, \frac{\alpha}{a}, n_3\right)$.

Step 2:

Using the second and third equations from the system of the first order conditions [(A.3.13) and (A.3.14)], solve numerically for m and n.

Step 3:

Using the results of step 2 (estimated values of m and n) and the fact that $A\left(z, \frac{\alpha}{a}, n_3\right)$ has $\frac{\alpha}{a}$ as an isolated term, we can derive the new values for $\frac{\overset{\curlywedge}{\alpha}}{a}$ from the first order condition with respect to n

$$\left(\frac{\overset{\curlywedge}{\alpha}}{a}\right)_1 = \ln m + \frac{n_2}{n_3}\frac{d\ln\Gamma\left(m, n\frac{\alpha}{a}\right)}{dn} + \frac{n_1}{n_3}\frac{d\ln\gamma\left(m+x+b, n\frac{\alpha}{a}\right)}{dn} - \frac{1}{n_3}\sum_{i=1}^{n_3} z_i.$$

If this value of $\frac{\overset{\curlywedge}{\alpha}}{a}$ is smaller than its previous value, the iteration is finished. Otherwise, the iterations must be continuous, i.e., starting with Step 2.

The algorithm produces $\left(\frac{\overset{\curlywedge}{\alpha}}{a}\right)_{t+1}$ from $\left(\frac{\overset{\curlywedge}{\alpha}}{a}\right)_t$ (where t is the number of iterations). The distance between the neighboring estimators,

$$\left|\left(\frac{\overset{\curlywedge}{\alpha}}{a}\right)_{t+1} - \left(\frac{\overset{\curlywedge}{\alpha}}{a}\right)_t\right| = \ln\frac{m_{t+1}}{m_t} + \frac{n_2}{n_3}\frac{d\ln\frac{\Gamma\left(m_{t+1}, n_{t+1}\frac{\alpha}{a}\right)}{\Gamma\left(m_t, n_t\frac{\alpha}{a}\right)}}{dn} + \frac{n_1}{n_3}\frac{d\ln\frac{\gamma\left(m_{t+1}+x+b, n\frac{\alpha}{a}\right)}{\gamma\left(m_{t1}+x+b, n\frac{\alpha}{a}\right)}}{dn},$$

determines the criterion for convergence of the parameter to its true value. Iterations are stopped when the last expression becomes less than $\varepsilon > 0$.

This section contains the econometric technique which enables us to estimate the distribution parameters m, n, and $\frac{\alpha}{a}$ through observing the random variable $\left(\frac{\alpha}{a} - \frac{p*\beta_i}{a}\right)$ from the censored sample.

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