Coase's conjecture in finite horizon

Michal Ostatnický

CERGE-EI

Charles University
Center for Economic Research and Graduate Education
Academy of Sciences of the Czech Republic
Economics Institute

WORKING PAPER SERIES (ISSN 1211-3298)
Electronic Version

241

Coase's Conjecture in Finite Horizon*

Michal Ostatnický

November 1, 2004

Abstract: In this paper Coase's Conjecture is analyzed in a finite-horizon formulation. In addition to utility discounting models decreasing-willingness-to-pay models are analyzed. We find that in contrast to Coase's Conjecture a monopolist may extract full monopoly profit in the finite-horizon problem under certain conditions; in fact, the monopolist does not have any reason to attract traders and waits until they come and trade. However, including utility discounting or decreasing-willingness-to-pay on the purchasers' side the monopolist's profit may dramatically decrease. The monopolist tries to clear trades as soon as possible, which makes him sacrifice a part of his one-shot monopoly profit to attract traders to buy.

Abstrakt: V tomto článku analyzujeme Coasovu hypotézu na modelech s konečným horizontem. Vedle modelů s diskontováním užitku analyzujeme též modely se snižující se ochotě platit. Zjišťujeme, že na rozdíl od Coasovy hypotézy může monopol v modelech s konečným horizontem získat celý monopolistický profit. Pokud do modelů zahrneme i diskontování užitku či snižující se ochotu platit na straně nakupujících, profit monopolu se může dramaticky snížit.

Keywords: Coase's conjecture, monopoly, discounting, decreasing willingness to pay.

JEL classification: C72, D42, G14, L11, L12.

*The author would like to thank Ronald Anderson, Randall Filer, Jan Hanousek, Andreas Ortmann, and Avner Shaked for helpful comments. This research was supported by the Grant Agency of the Czech Republic grant No. 402/04/0874.

I. Introduction

Coase(1972) showed a durable good monopolistic seller cannot extract monopoly profits in the infinite-horizon problem because he has to sell the good for a competitive price. His conjecture was analyzed in detail by Stokey(1981), Gul et al.(1986), and Ausubel and Deneckere (1989) who confirm Coase's conclusion. A possible way out of the monopolist's zero-profit trap is renting (see e.g. Bulow, 1982), which means a switch from Bertrand to Cournot competition. All of these papers discuss the infinite-horizon problem. Only Bagnoli et al. (1989) considered the finite-horizon problem in the form of several examples. In this paper we make the first attempt to analyze finite-horizon problems systematically. The analysis is motivated mainly by trading in stock markets, and the restriction on the interval when a stock market is opened during a trading day, therefore we deal with a finite-horizon problem.

The paper is split into two sections according to the behavior of buyers that want to buy a widget[1] from a monopolist. They may have either personal valuation of a widget and the utility obtained from the widget decreases over time as in the original setup analyzed in the mentioned literature (discounting), or the personal valuation itself may decrease over time (decreasing-willingness-to-pay). The decreasing-willingness-to-pay (WTP) may be applied to financial markets where traders narrow down the interval of prices they want to buy or sell a stock for.

The two sections are structured in the same manner with their propositions coinciding.[2] In both sections we find equilibrium strategies for the combination

[1] In Coase's paper they buy a piece of land as a symbol of durability. The fact that there is limited land area in the U.S. together with demand schedule determine the competitive price. The area size can alter the competitive price, not the general result. Even if the area is unbounded the competitive price would be zero or close to zero and the monopolist would charge the competitive price.

[2] See Appendix 1.A for the graphical structure of the paper.

of the following features: how buyers show up at the market (they can be present at the market from the beginning or they can gradually come there) and if they are patient or impatient.[3] Only the-willingness-to-pay model with gradual inflow of impatient traders has two versions with different results. Two different types of equilibria are shown to exist. Simpler models, especially the models with no discounting, lead to no trading equilibria during all periods but the last one when the one-shot game monopoly price is set and the monopolistic quantity is traded. The second equilibrium type results from discounting models. The price decreases during the day and transactions occur from the beginning of trading until the last period. Consumer surplus and social welfare increases when agents discount future profits compared to time patience.

The key differences between our conclusions and Coase's conjecture are the finiteness of the model and the continuous inflow of new buyers. The finiteness ensures all players that in the last period the monopolistic price will be set. The monopolist thus competes with himself in the future, which decreases his one-shot game profit with one exception: if nobody discounts future profits the monopolist waits to the last period to capture the one-shot game monopolistic profit. The continuous inflow of new traders prevent trading from occurring "in the twinkling of an eye" as Coase argued.

The paper is organized as follows, in the second section discounting models are analyzed, in the third section decreasing willingness to pay is introduced, and the fourth section concludes. The structure of propositions can be found in Appendix 1.A, proofs are listed in appendices 2 and 3.

[3]Patiency/impatiency is reflected in time discount or decreasing willingness to pay.

II. Utility Discounting

In this section the first type of analysis is carried out, in which it is assumed that agents discount utility from future trades and private valuations (or reservation price) themselves do not change. The models analyzed here can be directly compared to Coase's, Stockey's, Gul's, etc. analysis and the effect of the final period can be derived. The results range from totally different results (the monopolist can gain full-one shot monopolist profit) to nearly identical results (the monopolist's profit approaches zero as the number of periods, or the length of the day, increases). The profitability of waiting to the last period is the basic factor influencing the general result of the model. If traders do not discount future payoffs the monopolist profits from waiting to the last period when everybody who wants positive utility needs to trade. In the case of high discounts, waiting is too expensive for the monopolist and he is forced to trade as early as possible. The high discounts decrease substantially the monopolist's profit, and they bring the present model closer to Coase's model.

Imagine a market for a stock with one monopolistic market maker that has to quote the stock, and let us concentrate on bid quotes.[4] Assume that the traders appear on the market today and after the market closes they disappear and tomorrow the market maker faces completely different traders. This assumption allows us to separate trading days and to look at each one as a separate independent day with the given end: the end of trading. We can ask what is the optimal price path for the monopolist to get the highest possible profit. It definitely depends on the patience of the traders. If they are ready to wait until the end of trading the price path is different from a highly impatient traders market. Also, if the traders

[4]Setting a bid quote the monopolist (the market maker) offers to sell a given amount of shares. If we solve the bid quotes side then the ask quotes problem becomes symmetric.

come early in the morning with all of them willing to trade immediately at the beginning of the trading day the decisions of the monopolist will differ from the case when traders decide during the day to come and to trade.

The four models analyzed in this section are classified by traders' arrival at the market and their time patience/impatience to analyze the monopolist's behavior as described before. The basic structure of the market in the four cases remains unchanged. To keep the conclusions as general as possible let us analyze a market for widgets and their monopolistic producer. In our example, the stock can be viewed as a widget and the monopolistic market maker as the monopolistic producer.

Let there be a market for widgets. There is one monopolist that produces widgets and that can sell widgets on the market. The fixed and variable costs of production are zero, so that the monopolist profit is derived just from prices and concluded trades. The market is opened from time $t = 0$ to $t = 1$, when it is closed forever and it is not possible to transact any widgets.[5] On the market the monopolist posts prices in every instant during the trading day and he sells a widget to any customer that is interested in buying one.

Traders have a private valuation of a widget that is different for each trader and is drawn from a distribution that is common to all traders. Each trader has use of one widget only. Traders wait for the most appropriate moment to gain the highest possible utility from trading with the monopolist. Based on the arrival time of traders at the market we distinguish two models. First, all traders come to the market during the night before the market opens. From their private valuations we can construct a limit order book[6] that the monopolist trades against during

[5] This makes the model a finite-horizon problem and different from Coase's conjecture.

[6] The limit order book can be viewed as a demand schedule, however, it is not a demand *function* as such. The individual valuations are not prices that traders are willing to accept any time. The traders behave strategically so we call this 'demand schedule' a limit order book.

the trading day. Second, there is nobody on the market at $t = 0$ and traders arrive during the trading day. The monopolist has to set prices to gain the maximum profit as in the first case. In real markets, the two models are mixed up, there is a limit order book at the beginning of the day and additional traders come to the market throughout the day. Such a real market behaves as in the first model and then slowly changes to the behavior of the second model as the influence of the initial limit order book decays and more and more fresh traders appear.

Initial-order-book models

As mentioned above, N traders come at the market during the night.[7] Every trader k has a private valuation v^k of a share that is drawn independently for each trader from a distribution with cumulative distribution function $\Gamma(\cdot) = Prob(v^k \leq \cdot)$.[8]

To make the modelling simpler let the trading day [0,1] be split into T discrete periods. At the beginning of every period t the market maker has to post price p_t, thereafter every trader present on the market decides whether to trade at the given price. The liquidity traders that have traded leave the market. All trades are immediately observable. The traders maximize their net utility, which is the difference between their private valuation and the trade price,

$$U^k = v^k - p_t,$$

where t is the period of trade execution. Traders have an outside option not to trade that gives them zero profit. They have no time preference (they have zero discount factor waiting for a better price) and perfect foresight and thus wait for

[7]Between the close of the previous trading day and the start of a new trading day ($t = 0$).

[8]For example, in the stock markets the traders can represent noise traders (as used by e.g. Kyle, 1989, Kyle and Vila, 1991). These noise traders have private valuations drawn from a normal distribution. The mean of the distribution represents the mean of all private valuations. If there is no private information pending to be used on the market the mean represents the (discounted) future stock value.

the best price they can acquire. The monopolist maximizes the expected profit that is the sum of the profits from each trade:

$$\Pi^M = \sum_{t=1}^{T} NS_t * p_t,$$

where NS_t is the number of sales in the period t.

The strategy of traders in every period is the decision to trade or not to trade based on the current prices, time period, previous transactions, and expectations of future prices. The strategy of a buyer k in every period $1 \leq t < T$ is to purchase a widget or to stay in the market. In the last period the strategy is to buy or to get the outside option of zero profit. The strategy of the monopolist is to set in every period the price based on past trading and expectations of future trading. The expectations are based on the knowledge of statistical parameters of the demand (the function Γ and the expected number of traders coming N) that is publicly available. The following lemma describes the role of the parameter N, i.e. the expected number of traders coming to the market.

Lemma 2.1

The number of traders does not influence the price path. The market maker behaves as if the first period demand function were $1 - \Gamma(p)$.

Proof: see Appendix 2. •

The function $N * (1 - \Gamma(p))$ represents the expected initial order book that the monopolist faces. Because he does not see the real order book, he needs to set the quotes as if there were numerous traders with valuations copying the distribution function as the lemma states. He also does not get any additional information about the actual realization of the random variables v_k due to the independence of the valuations.

Proposition 2.2

Assume the individual valuations distribution $\Gamma(\cdot)$ is continuous and the function $(1 - \Gamma(p))p$ has only one maximum. All traders are present on the market from the beginning and they are patient (they have zero discount factor). In any Nash equilibrium in pure or mixed strategies the last period quotes are set to monopolistic prices and all transactions take place just in the last period. In the previous periods no trade is concluded in pure strategy equilibrium. •

The core idea is presented here. Detailed proof can be found in Appendix 2. The monopolist sets prices with respect to the residual demand in the last period. There is no future after period T so everybody with higher valuation than the price p_t trades, and this enables the monopolist to set prices at monopoly levels. The traders have perfect foresight and no time preference so they do not want to buy for any price higher than the price in the last period. The monopolist has also perfect foresight so he would not set the bid lower than the monopoly price because everybody would trade immediately, and the monopolist's profit would be smaller. Even the monopoly prices cannot push the traders to trade during the day. Such a transaction would necessarily decrease the expected demand and push down the monopoly price. Because of perfect foresight these traders would make a mistake. The mechanism is explained on a linear demand function in Appendix 1.B.

As an example, consider normally distributed individual valuations with the distribution function $\Gamma \equiv N(0, \sigma)$[9] and compute the monopoly prices and the welfare loss. The monopolist maximizes the last period profit $p_T\Gamma(p_T)$, which is equal approximately to $p_T \doteq 0.75\sigma_U$. The result can be summarized as a proposition.

[9]There are many factors influencing the individual valuation, so the normality of the distribution is a natural assumption. We may also WOLOG assume the individual valuations to be negative, these traders simply use the outside option of zero utility in the last period and the model is equivalent to one with a normal distribution censored at 0.

Proposition 2.3

$$p_T^M \doteq 0.75179\sigma$$

The welfare loss is

$$\Delta W \doteq 0.2\frac{N}{\sigma},$$

where $\phi(\cdot)$ represents the partial distribution function of normal distribution. The welfare loss is linearly increasing with the number of traders and it is decreasing with the volatility of individual valuations. •

The model above is solved for the case of unlimited amount of widgets offered every period. If the monopolist also has to set the depth, i.e. the number of widgets for which the price is valid, the solution remains the same. We have shown above that the monopolist sets the price to be a monopoly price with respect to the expected demand in the last period, thus he does not have any reason to restrict the quote depth. The more widgets he sells the higher profit he gets, and because there is no future, he cannot influence any future profits by selling the maximum amount of widgets. Moreover, the monopolist cannot commit to the last period price and volume traded, thus he cannot influence even the previous periods sales.[10] Using backward induction we can show the quotes are unrestricted even in previous periods. Nevertheless, there is trading activity only in the last period. The quotes are always valid for an unrestricted amount of shares even if the restriction on the volume offered is permitted.

The market maker we talked about at the beginning of this section sets the bid quote to the monopolistic price with respect to the expected demand and unlimited quote depth.[11] Everybody waits until the very end of the trading day

[10] If he could we would switch to restricted capacity models.

[11] Here we assume that the market maker knows the shape of the Γ distribution function. We can, for example, expect the distribution function to be historically stable so it is publicly known.

when somebody trades. Because the price and quantity traded equals one-shot monopoly levels, the welfare loss is quite big and we may want to regulate the market and impose some restrictions on the bid-ask spread. Usually, the traders are impatient and they discount the future payoffs. The optimal price path changes and its shape depends on the discount factor of the market maker himself. The next model describes what happens, in equilibrium, when traders discount future profits.

Assume the model stays the same as before, the monopolist has to set the price of a widget in every instant, traders are accumulated on the market in the period $t = 0$, they have individual valuations of a share drawn independently from the distribution Γ and they have perfect foresight. Assume the traders have a period-to-period discount factor δ, so that the utility of trader k gained from a transactions in period t is $U^k = (1 - \delta)^{t-1}(v^k - p_t)$. The market then behaves in a completely different way.

Proposition 2.4

Assume the traders discount the future by a factor $\delta > 0$ so that personal valuation in period t satisfies $U_k = (1-\delta)^{t-1}(v^k - p_t)$, the monopolist also discounts the future by the factor β. Assume the profit function $p(1 - \Gamma(p))$ is concave in interval $p \in (0, p^M)$. Then there is a unique price path and strategies that constitute a Nash equilibrium. The quantity of shares traded in each period is equal to the monopoly quantity with respect to the current-period residual demand if the discount factors δ and β coincide; it is greater than the one-shot monopoly price with respect to the residual demand if $\beta > \delta$ and vice versa. The prices are lower than the monopoly prices in periods $t \in \{0, \ldots, T-1\}$. The welfare loss is smaller than for $\delta = 0$. •

The proof can be found in Appendix 1.A. You can also see there that the last period price is a monopoly price with respect to the residual demand, and in every previous period t the bid quotes satisfy $\frac{p_t-(1-\delta)p_{t+1}}{\delta} = p_t^M$, where p_t^M is the monopoly price with respect to the t period residual demand. The immediate consequence is the fact that the bid quotes monotonically decrease and ask quotes monotonically increase to the competitive price. Also trading activity is highest in the first period and monotonically goes down.

Compare this result also to the case when the monopolist can set prices in advance. In such a case he sets the monopoly price in the first period and bigger than this price in the following periods gaining one shot game monopoly profits. Still he is not able to appropriate the whole traders' surplus (see Appendix 1.B for a precise description of this model).

Traders-coming-throughout-the-day models

It is a rare phenomenon that the order book is full at the beginning of trading and no new orders come during the trading day. Our market maker waits for traders possibly entering the market and sets quotes for them. He knows approximately how often traders come to the market (this is not an essential assumption right now) and the distribution of their private valuation. Does the price path vary or does the market maker actively search for the actual realizations of private valuations? Even in this case the market maker sets one-shot monopoly quote and gets the a one-shot monopoly profit.

Let us assume the traders' and the monopolist's characterizations are the same as in the first model. The monopolist has to set the price every period, based on this price traders decide to conclude a trade or to stay in the market. The traders have individual valuations, and they have no time preference and perfect foresight. They do not come to the market before opening; they appear there one by one during the trading day. It is not essential for the model exactly when traders come. The resulting equilibrium price path does not differ from the initial-order-book model with patient traders.

Proposition 2.5

In any Nash equilibrium in pure or mixed strategies the prices are set to one period monopoly level or higher in every period, and to one-shot monopoly level in the last period. The traders trade just in the last period. •

The proof can be found in Appendix 2. Its idea is essentially the same as the proof of Proposition 2.4.

We can see now that the monopolist's position is clear when traders are patient and do not discount future profits. He waits until the last period, lets the traders accumulate in the traders-coming-throughout-the-day model and waits, and at the end of the trading day he gets the one-shot monopoly profit. As in the previous type model we expect that discounting may shift some of the power of the monopoly to the traders.

Assume that traders' utility is the difference between their individual valuation and the price paid for a widget discounted by factor δ per period waiting and zero if they do not trade. The monopolist has a similar type of utility function. He discounts future trades by factor β. in the same way as traders do. No trader is present on the market before opening. Each period the same amount of agents come to the market.[12] The monopolist has to optimize his price-path to maximize profit. The result is the same as in Proposition 2.4. Both propositions work with current demand - even if the conclusion is the same the resulting equilibrium price path differs.

[12] If we assume traders come to the market according to the Poisson arrival rate the probability that somebody is willing to trade for price p is $\lambda\Gamma(p)$ in the first period, where λ is one period arrival rate. The same increase in the actual demand function can be observed every other period.

Proposition 2.6

Assume the traders discount the future by a factor $\delta > 0$ so that personal valuation in period t satisfies $U_k = (1-\delta)^{t-1}(v^k - p_t)$, and they come to the market in groups of λ traders each period. The monopolist also discounts the future by factor β. Assume the profit function $p(1-\Gamma(p))$ is concave on the interval $p \in (0, p^M)$. Then there is a unique price path and strategies that constitute a Nash equilibrium. The quantity of shares traded in each period is equal to the monopoly quantity with respect to the current period residual demand if the discount factors δ and β coincide, and is greater than the one-shot monopoly price with respect to the residual demand if $\beta > \delta$ and vice versa. The prices are lower than the monopoly prices in periods $t \in \{0, \ldots, T-1\}$. The welfare loss is smaller than for $\delta = 0$. •

There is a different way of how to introduce a discount rate. In contrast to the discounting of future profits it is possible to assume that the individual valuation itself changes over time. There must be a reason to have an individual valuation different from others, and this reason can change over time. For example, on a stock market traders can find themselves to be too optimistic/pessimistic and they reassess their private valuation. This drives the private valuations towards the mean, i.e. the spread of individual valuations decreases over time. For now, let just the trades that come to the market in the middle of the trading day have a smaller spread of individual valuations than those coming earlier.[13] Let us assume that every period the same amount of traders come to the market.[14] Every period the spread of individual values decreases, i.e. their valuations are drawn from a distribution $\Gamma(\cdot, \sigma_t)$, where $\sigma_t > \sigma_{t+1}$. This causes the monopoly prices

[13] If all traders reassess the individual valuation we would be in the situation of decreasing willingness-to-pay models. These are analyzed in the next section.

[14] For example, if the arrival rate follows a Poisson process the expected amount of traders coming to the market is the same for every period.

with respect to current period newcomers to monotonically decrease/increase to the average value. Even if there is such a discount rate, the resulting equilibrium resembles the previous model.

Proposition 2.7

In any Nash equilibrium in pure or mixed strategies the last period prices are set to the total demand $\sum_{t=0}^{t=T}(1-\Gamma(p,\sigma_t))$ monopoly level. There are no trades concluded prior to the last period. •

The reason for no trades prior to the last period stays the same as in the first model. Because the monopoly quotes with respect to the expected cumulative demand decrease and traders are time patient they wait for better prices. Even if the monopolist sets the prices during the day equal to the last period quotes, nobody would trade because they would depress the monopoly prices.

In reality, there exist typically traders at the beginning of a trading day and other traders who come to the market throughout the day. We have to combine previous model types to get the equilibrium trading volume and price paths. The accumulated order book pushes the monopolist to behave according to the first model type, if the traders have some time preference the market maker tries to make profit at the expense of the traders that came to the market during the night. During the day, as these traders leave the market and new traders come, the strategy of the market maker changes to the second type model. This results in the heaviest trading taking place at the beginning and at the and of the trading day. This result is in line with the U-shaped trading quantity phenomenon observed,for example, on stock markets (see e.g. Niemeyer and Sandas, 1993).

III. Decreasing Willingness-to-Pay

We can think of a different kind of discounting on the purchaser's side. Assume that the utility gained from a widget declines over time for every trader because he is willing to pay less and less as time goes on. In fact, this means that the private valuation itself decreases over time. Let us look at the market we use as an example. There are a monopolist market maker and traders willing to buy a stock on the market. Here it is worthwhile to suppose the market maker knows the common value of the share and the traders have their private valuations distributed around the common value. The following scenario may be possible: the traders, observing the prices on the market, can reassess their investment plans and find themselves being too optimistic about the common value so that their private values decrease. Symmetrically, the pessimistic traders can increase their private valuations. Both of these effects lead to the convergence of private valuations towards the common value. Moreover, traders present on the market may find some other investment opportunities so they are reluctant to buy at high prices later. These features can be described as decreasing willingness-to-pay. It differs from discounting so that utility from a purchase in time t in the decreasing willingness-to-pay by factor δ case is

$$U = (1-\delta)^{t-1}v^k - p_t,$$

where v^k is the individual k's first period individual valuation of a widget.[15]

The decreasing willingness-to-pay cannot be simply transformed into utility discounting. From the point of view of a trader nothing changes; if he trades the utility from a transaction is discounted. On the other hand, the monopolist perceives the situation differently. For example, imagine a situation when buyers

[15] The utility in the discounting case is $U = (1-\delta)^{t-1}v^k - p_t$.

discount their utility from future transactions. Nobody trades until the last period because the monopolist knows that the buyer's private valuations stay unchanged from the beginning of the trading day, so he charges them the one-shot monopoly price that stays constant throughout the day. Suppose these traders' willingness-to-pay decreases. The one-shot monopoly price decreases as times goes on and the monopolist gains much lower profit than in the case of utility discounting. The surplus is split between both, the monopolist and the buyers, in contrast to the utility discounting when the monopolist captures the whole surplus.

This version of 'discounting' is valid mainly in markets sensitive to the information available. If my valuation depends also on general information that is gradually revealed I can restructure my beliefs and change my individual valuation. Such improvements of individual valuations lead to a decreasing spread of valuations and convergence to one common valuation with some disturbances. During this process traders increase their individual valuation if they are under the common value and they decrease it if they are above. Because the monopolist (or our market maker) does not want to sell under the common value, the buyers with individual valuation under the common value have no chance to trade, therefore they drop out from the analysis. We also assume in this section that the common value is 0.

As described before the decreasing willingness-to-pay problem is similar to the previous section discounting problem. We follow the same structure as in the previous section, but the results may be slightly different here. Looking at the utility functions we can immediately observe that for $\delta = 0$ the two problems completely coincide thus propositions 2.2, 2.5, and 2.7 remain formally unchanged. Propositions 2.1 and 2.3 stay valid even for decreasing willingness-to-pay.

Initial order book models

As in the previous chapter we start with the model of all traders being present on the market from the beginning of the trading period. Traders maximize utility $U = (1-\delta)^{t-1}v^k - p_t$ and the willingness to pay does not decrease over time, so the δ term vanishes. The monopolist has to set the price of a widget every period. He maximizes the cumulative profit $\Pi^M = \sum_{t=1}^{T} NS_t * p_t$, where NS_t is the number of sales in the period t. Because $\delta = 0$ the model degenerates to the first model of the previous chapter. We can formulate a proposition similar to Proposition 2.2.

<u>Proposition 3.1</u>

Assume the individual valuations distribution $\Gamma(\cdot)$ is continuous and the function $(1-\Gamma(p))p$ has only one maximum. All transactions are present on the market from the beginning and their willingness to pay does not decrease. In any Nash equilibrium in pure or mixed strategies the last period quotes are set to monopolistic prices and all trades take place just in the last period. In the previous periods no trade is concluded in pure strategy equilibrium. •

The proof is identical to the proof of Proposition 2.2 so we refer the reader to that one.

Now we introduce the new type of discounting, the decreasing willingness-to-pay. As before we assume that traders maximize their utility $U = (1-\delta)^{t-1}v^k - p_t$, δ is now positive. It means that individual valuation decreases over time reflecting the decreasing willingness to pay. The decrease is relative and constant. Traders are present at the market from the beginning, they have perfect foresight, and their utility is independently drawn from the distribution function $\Gamma(\cdot)$. The monopolist maximizes cumulative profit $\Pi^M = \sum_{t=1}^{T} NS_t * p_t$. We do not introduce any discount factor for the monopolist because traders also do not have any. A model with decreasing willingness to pay and positive discount factor can be easily derived in line with this model and the respective model of section 2. The decreasing willingness-to-pay model without discounting leads to a simple result: the quantity traded is always equal to the one shot monopolistic quantity with respect to residual demand and the prices are depressed towards zero. We can summarize these facts in the following proposition.

Proposition 3.2

Assume the traders' willingness to pay decreases by a factor $\delta > 0$ so that personal valuation in period t satisfies $U_k = (1-\delta)^{t-1}v^k - p_t$. The monopolist maximizes undiscounted cumulative profit $\Pi^M = \sum_{t=1}^{T} NS_t * p_t$. Assume the profit function $p(1-\Gamma(p))$ is concave on the interval $p \in (0, p^M)$, where p^M is one shot monopolistic profit with respect to $\Gamma(\cdot)$. There is then a unique price path and strategies that constitute a Nash equilibrium. The quantity of shares traded in each period is equal to the monopoly quantity with respect to the current period residual demand. The prices are lower than the one shot monopoly prices in periods $t \in \{0, \ldots, T-1\}$. The welfare loss is smaller than for $\delta = 0$. •

We point out here that the result of the monopolistic quantity with respect to the residual demand traded does not depend on the value of the decreasing willingness-to-pay factor. The quantity traded depends solely on the actual demand.[16] This result, on the other hand, gives the model the same drawback as Coase's original conjecture: if we shorten the time intervals the number of periods when the quantity is sold increases. The price is driven down as the number of periods increases and is equal to the competitive (zero) price in the limit of infinitely short periods.[17]

[16]It depends also on the discount factor if we introduce impatience in the model.

[17]In fact, in the limit the model is similar to Coase's model as the infinitely short time periods are equivalent to the infinitely long trading day.

Traders coming sequentially

In standard markets a fast decline in prices and a huge amounts of sales are not observed because newcomers improve the situation. The inflow of new traders with characteristics similar to old traders to some extent 'restart' the market. We can classify two model types as before. Newcomers can start with completely the same characteristics as older traders had when they came to the market, i.e. homogeneous traders, or, alternatively, the newcomers can start with similar characteristics as older traders currently have in the period they come. Rephrasing this in the terms of the previously described models, the private valuation of every newcomer can be drawn from the original distribution $\Gamma(\cdot)$ as in the first case, or from distribution $\Gamma(\cdot/(1-\delta)^t)$. The first case does not fit the reality if we think about the decreasing willingness-to-pay as of the situation when common information outweighs private valuations. As for the depreciation the homogeneous case seems to be more meaningful, the traders' utility depreciates from the moment they want to trade something. The decreasing willingness-to-pay model makes more sense when newcomers start with adjusted individual valuation.

The equilibria of the two models are summarized in the two following propositions. There is no surprising result. The homogeneous traders model results in one shot monopolistic quantity with respect to residual demand. The gradual inflow of customers alters just the residual demand, not the key result directly. The second proposition presents the equilibrium of the adjusted newcomers' characteristics model. The model is similar to its depreciation counterpart and the result is completely the same.

Proposition 3.3

Assume the decrease factor in a decreasing willingness-to-pay model is $\delta > 0$. Personal valuation *in period l of a trader k's life* satisfies $U_k = (1-\delta)^{l-1}v^k - p_t$, and traders come to the market every period in the groups of λ. The individual valuations v_k are independently drawn from the distribution $\Gamma(\cdot)$. The monopolist does not have any discount factor. Assume the profit function $p(1-\Gamma(p))$ is concave in the interval $p \in (0, p^M)$. There is then a unique price path and strategies that constitute a Nash equilibrium. The quantity of shares traded in each period is equal to the monopoly quantity with respect to the current period residual demand. •

Proposition 3.4

Assume the decrease factor in a decreasing willingness-to-pay model is $\delta > 0$. Assume the personal valuation of the trader k *in period* t satisfies $U_k = (1-\delta)^{t-1}v^k - p_t$, and traders come to the market in groups of size λ. The monopolist does not have any discount factor. Assume also that the individual valuations v_k are independently drawn from the distribution $\Gamma(\cdot)$. Then in any Nash equilibrium in pure or mixed strategies the last period prices are set to the total demand $\sum_{t=0}^{t=T}(1-\Gamma(p,\sigma_t))$ monopoly level. There are no transactions prior to the last period. •

The proof of Proposition 3.3 can be found in Appendix 3, the proof of Proposition 3.4 is identical to the proof of Proposition 2.7.

The previous types of models can also be combined. If we combine the models we call "more meaningful", e.g. homogeneous traders in discounting and decreasing willingness-to-pay over all traders, including those not yet started, we get even closer to the standard (stock) market setup. Such a model may result in a conclusion similar to Proposition 2.6 with the discount rate equal to the average of the two discounts used in "meaningful" models.

IV. Conclusion

In this paper we give an overview of the monopolist selling in finite horizon problem. The idea developed by Coase(1972), however, has been analyzed in its entirety in the domain of infinite horizon. In this paper the monopolist is allowed to set the price freely in the trading period, but it obliges him to sell any amount of widgets that is demanded (posted prices). The monopolist's target is to exploit the maximal share of surplus created by the variability of traders individual valuations.

We distinguish three basic features of the model. First, the traders may be impatient, i.e. they may discount future profits, or their willingness to pay may decrease over time. The impatience assumption is standard, the decreasing-willingness-to-pay may be caused by common information being revealed to all participants so that the 'optimistic' ones come closer to the competitive level or the purchase of a product becomes less and less profitable. Second, all traders can be present at the market at the beginning of the trading day or they can come to the market sequentially. The real markets appear to be a mixture of the two models, some traders come to the market immediately after the opening and some come later on. One of these features, nevertheless, dominates and we can say that the market is closer to the equilibrium of either one or the other models. Third, the discount factor can be reduced to zero, i.e. we deal with complete patience or stable willingness to pay.

The resulting equilibria have several general features.[18] The zero discount factor or the stable willingness to pay lead to an equilibrium where all transactions are concluded in the last period and the price is set to the monopolistic price with respect to the residual demand. This result holds irrespective of the coming to the market feature. The same equilibrium is found also in the cases when the traders

[18] A complete overview of models' equilibria is given in Appendix 1.

come sequentially to the market and the newcomers replicate the characteristics of traders present at the market. The equilibrium includes trading in all periods in the case of the positive discount/decreasing willingness-to-pay factor assuming traders being present at the market from the beginning, or traders coming sequentially with newcomers having homogeneous characteristics when they start. The quantity traded is equal to one shot game monopolistic quantity with respect to the current residual demand. Prices are set above the competitive level, but well under the monopolistic prices.

We can derive also a wide range of mixed models: either the mixture of traders present at the market from the beginning and traders coming later on or the mixture of discounting and decreasing willingness to pay. These models may describe real markets, e.g. stock markets and the behavior of traders and sellers/market makers. The models are derived for the monopoly case. Competition between several sellers may decrease prices or it may not. Using the presented model we can study, for example, implicit collusion on markets comparing the price path in a market with the monopolistic model. Another application of the model can be found in the market regulation domain. We can derive theoretically the equilibrium price and monopolist's profits on a market and find out if the profits cover the selling costs of the monopolists. If the distribution of private valuations is tight, close to the competitive price, the monopolist can make just a small direct profit from trading and in the case of big selling costs he would not be willing to participate at the market. Direct application of this feature can be found on security markets and monopolistic market making. The models presented here can be extended to the oligopoly case or competitive market case, but such models are beyond the scope of this paper.

References

Ausubel, L.M., and R.J. Deneckere, 1989. "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica* 57(3), 511-531.

Bagnoli, M., S.W. Salant, and J.E. Swierzbinski, 1989. "Durable-Goods Monopoly with Discrete Demand," *The Journal of Political Economy* 97(6), 1459-1478.

Bulow, J.I., 1982. "Durable-Goods Monopolists," *The Journal of Political Economy* 90(2), 314-332.

Coase, R.H., 1972. "Durability and Monopoly," *Journal of Law and Economics* 15(1), 143-49.

Gul, F., H. Sonnenschein, and R. Wilson, 1986. "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory* 39, 155-190.

Kyle, A.S., 1989. "Informed speculation with imperfect competition," *Review of Economic Studies* 56(3), 317-355.

Kyle, A.S., and J.L. Vila, 1991. "Noise trading and takeovers," *Rand Journal of Economics* 22(1), 54-71.

Niemeyer, S., and P. Sandas 1993. "An Empirical Analysis of the Trading Structure at the Stockholm Stock Exchange," *Journal of Multinational Financial Management* 3 (3-4), 63-101.

Stockey, N.L., 1981. "Rational Expectations and Durable Goods Pricing," *The Bell Journal of Economics* 12(1), 112-129.

Appendix 1.A

The Structure of the Paper

	Utility discounting		Decreasing willingness to pay	
	no discounting	discounting	no discount	discount
Initial order book models	Proposition 2.2	Proposition 2.4	Proposition 3.1	Proposition 3.2
Traders sequentially coming	Proposition 2.5	Propositions 2.6, 2.7		Propositions 3.3, 3.4

Appendix 1.B

The Simplest Game with a Linear Inverse Demand Function

To see the equilibrium we analyze the case of a linear inverse demand function in this appendix. Let there be a monopolist producing a good with zero marginal cost and facing linear inverse demand function $Q(p) = a - bp$. The monopolist can sell the good in two periods. Those customers that buy the good in the first period immediately leave the market and the monopolist charges the residual demand monopoly price to the rest of the buyers. In this case the monopoly price of a one shot game is $p^M = \frac{a}{2b}$, so for the two-period game nobody will buy the good for a price $p \geq p^M$, if the monopolist charges such a price everybody would wait to the second period to pay p^M. Also, there is no equilibrium in pure strategies in the case $p_1 < p^M$. If this equilibrium is to stay in the market (buy in the second period), the buyers make a mistake because they buy for $p^M > p_1$.

If the equilibrium is to buy in the first period, they also make a mistake because the second period monopoly price is $\frac{bp_1}{2b} = \frac{p_1}{2} < p_1$. This argument holds for any price. If the strategy is to buy for p_1 in period 1, the price in period 2 is $\frac{p_1}{2}$. Thus the strategy to buy in the first period is never an equilibrium strategy. There is just one equilibrium in pure strategies: the monopolist sets the first period price greater or equal to p^M in the first period, nobody buys, and then sets the second period price equal to p^M.

There may exist another equilibrium in mixed strategies. Let r be the probability that the buyers whose valuation of the good is greater than p_1 buy the good. The inverse demand function in the second period is

$$\begin{aligned} Q_2(p) &= a - r(a - bp_1) - ((1-r)b + \frac{b^2}{a} r p_1)p, \quad & p \geq p_1 \\ &= a - r(a - bp_1) - bp, & p < p_1 \end{aligned}$$

Because of the kink we have to solve two maximization problems. The problems for the part $p < p_1$ gives us a monopoly price in the second period

$$p_2 = \frac{(1-r)a + rbp_1}{2b}$$

if $p < p_1$, a solution does not exist otherwise. The solution of the part $p_2 \geq p_1$ must satisfy

$$p_2 = \frac{(1-r)a + rbp_1}{2((1-r)b + \frac{b^2}{a} r p_1)} \qquad (*)$$

if $p_2 \geq p_1$, a solution does not exist otherwise. Because all the buyers must be indifferent between buying in the first and in the second period the equality $p_1 = p_2$ must hold. We do not need an explicit formula for p, the monopolist maximizes profit setting the price to the one shot monopoly price. We have two kinds of equilibria. One, the monopolist sets the price bigger or equal to p^M in the first period and equal to p^M in the second period, traders buy just in the second period.

Second, in both periods the monopolist sets the price to be p^M in both periods, in the first period the fraction[19]

$$r^* = \frac{a - 2bp^*}{a + 3bp^* - 2\frac{b^2}{a}p^{*2}}$$

of buyers with higher valuation trades, in the second period the rest trades. If the value of r^* is positive, the market maker is forced to set the monopoly price even in the second period. Because $p^* = p^M$, we can substitute it into the previous equation and we get $r^* = 0$.

If the agents with different valuation are supposed to have strategies dependent on their valuation, there can arise another equilibrium. In such equilibrium the monopolist sets both period prices to be p^M, the reason is again that the traders must be indifferent between buying in both periods. It may be the case that just traders between p^M and $\frac{3}{2}p^M$ disappear in the first period. Does there such an equilibrium exist? The answer is yes. Suppose the fraction r of traders with valuation at least p^M buys in the first period. If we want to find the case when the second period monopoly price is the highest, the second period demand function $\overline{Q}(p)$ should be $a - bp - r(a - bp^M)$ for $p \leq p^M$, $a - bp$ for $p \geq (1 + r)p^M$ and constant in-between.[20]

The monopoly price with respect to the $\overline{Q}(p)$ demand function may be found in the two intervals: $p \leq p^M$ or $p \geq (1 + r)p^M$. As for the second interval, the demand function is the same as before, so we have a corner solution giving to the monopolist total profit of $\left(\frac{1+r}{2}\right)^2 \frac{a^2}{b}$. The left interval gives the monopoly price

$$p = \frac{(1 - r)a + rbp^M}{2b} = \frac{a(2 - r)}{4b} > \frac{a}{2b} = p^M,$$

the solution is just the corner solution $p_2 = p^M$ leading to the monopolist's profit $\frac{a^2}{4b}(1 - r)$. The later profit is smaller whenever $r > 0$ so there are many possible

[19] obtained from (*)

[20] For any other demand function $\widetilde{Q}(p)$ where $rQ(p^M)$ traders disappear in the first period $\widetilde{Q}(p) \leq \overline{Q}(p) \forall p$.

equilibria. Transactions of high valuation customers push the second period price down, transactions of low valuation customers push the second period price above p^M.

The multiple period problem can be solved in a similar manner. Assign the number of periods T and let indexed variables be the values of a specified period. There is no pure strategy equilibrium of the T-period problem, if any of the prices p_t is below the one shot game monopoly price p^M. The only set of pure strategy equilibria constitute prices $p_t \geq p^M, p_T = p^M$ and sellers' strategies not to buy in any period $t = 1, \ldots, T-1$ and to buy in the last period if the individual valuation is bigger[21] than p^M, not to buy otherwise.

For the mixed strategy equilibrium denote r_t the fraction of people valuing the good more than p_t that buy the good in the period t. The value $r_t = 0$ indicates there is no trade in the round t in equilibrium. Construct series $j_1, \ldots, j_n$ so that $r_{j_i} > 0, i = 1, \ldots, n$. First we show that $r_T = 1$. The market maker never charges the price 0 because then he would get zero profit. The number of periods is finite, the minimum of prices charged up to $T-1$ is bigger than 0. The inverse demand function is continuous so there is some outstanding demand in the period T, thus the monopoly price in the last period is positive and $r_T = 1, j_n = T$.

The customers must be indifferent between buying in the period T and j_{n-1}, the price $\overline{p}$ charged in these two periods must be identical. The same holds for all periods $j_i, i = 1, \ldots, n$, in other periods the price must be bigger or equal. For the market maker to maximize his profit he has to set $\overline{p} = p^M$. In the period j_1 the fraction of r_{j_1} appropriate customers trade. This fraction must be such that it forces the market maker to set the monopoly price p^M. It was shown above for the two-period model that this fraction is $r_{j_1} = 0$. This is in contradiction with

[21] Or equal, depends how we want to define the behavior on the margin. The result remains the same in both cases

the construction of the $j_{i i=1}^{n}$ series if $n > 1$. It means that $n = 1$ and the trades are concluded just in the last period.

The equilibria then satisfy $p_t \geq p^M$ and traders with valuation greater than p^M purchase the good in the period T, there is no trade otherwise. There is, again, a set of equilibria where a customer with valuation greater or equal to p^M purchases the good in the period t that satisfies $p_t = p^M$ and the market maker adjusts the monopoly price from this period on.

Appendix 2
Proofs from the Section II

Proof of Lemma 2.1

We need to show that the shape of the current demand function is equivalent to the shape of the function $1 - \Gamma(p)$, where $\Gamma(p)$ is the distribution function of traders' individual valuations. Let us assume that the number of traders present in the market is known and it equals N. The expected demand function with N traders present, i.e. the expected number of traders that have higher individual valuation than the price p is thus equal to

$$Q^N(p) = \sum_{i=0}^{N} i \binom{N}{i} \Gamma^{N-i}(p) \left(1 - \Gamma(p)\right)^i .$$

The sum on the right hand side is the expected value of a binomial distribution with parameters $(N, 1 - \Gamma(p))$. The demand function can be written as

$$Q^N(p) = N(1 - \Gamma(p)).$$

The number of traders is a factor that scales the actual profit linearly up and down, the number of traders does not influence the decision about the optimal price p. The result is not influenced by the fact whether the number of traders N is known or unknown. *Q.E.D*

Proof of Proposition 2.2

The trading day has a last period T, the proposition is proven by a backward induction. Keep in mind that the initial period demand $Q_1(p)$ relates to the distribution function of individual valuations $\Gamma(p)$ so that $Q_1(p) = 1 - \Gamma(p)$. To follow backward induction we start to analyze the last period.

Period T

There is a residual demand $Q_T(p)$ in the last period.

Traders take any price they can make positive profit on because this is the last period and they would leave the market with zero utility if they do not take this price.

The market maker sets the quotes at monopoly prices with respect to the sell/buy expected demand because traders accept any price if they can make positive profit.

Period $T-1$

The traders consider the offered price p_{T-1}, expected demand $Q_{t-1}(p)$ and choose whether to trade or to wait for the next period. If the price offered is strictly smaller than the one shot monopoly price with respect to the current expected demand p^M_{t-1} then everybody would tend to buy. But buying would be a mistake. If everybody (in fact, if anybody) buys the period T monopoly price would be even lower. In that case the traders would have made a mistake buying in the period $T-1$ thus such a behavior is not a part of an equilibrium. Thus in pure strategy equilibrium the $T-1$ period price cannot be smaller than the monopoly price with respect to the current demand.

As for the mixed strategies, if a fraction q of traders buy (they use mixed strategies 'buy with probability q if you can make a positive profit.') they have to get the same profit as those staying to the last period (this requirement follows from mixed strategy conditions), thus the prices in the last two periods must equal $p_{T-1} = p_T$. By assumption p_{T-1} is smaller than the one shot monopoly price. In this case the market maker makes a mistake, he would get more setting $p_{T-1} \sim \infty, \quad p_T = p^M_{T-1} = p^M_T$. The price p_{T-1} cannot be strictly smaller than the current period monopoly price p^M_{T-1}.

Assume that the price p_{T-1} is higher than or equal to the monopoly price with respect to the current demand and the fraction $q > 0$ of traders with higher valuation than p_{T-1} buys. The related period T demand is

$$\begin{aligned} Q_T(p) &= Q_{T-1}(p) - qQ_{T-1}(p_{T-1}) \qquad p \leq p_{T-1} \\ &= (1-q)Q_{T-1}(p) \qquad\qquad\quad\; p \geq p_{T-1} \end{aligned}$$

The first order conditions of the profit maximization problem $MaxpQ_T(p)$ have to be split into two parts:

$$p > p_{T-1} \geq p^M_{T-1} \qquad (pQ_T(p))' = (1-q)Q'_{T-1}(p)p - (1-q)Q_{T-1}(p) < 0,$$

so the final period monopoly price cannot be higher than p_{T-1}. On the other side the following equation should be satisfied

$$p \leq p_{T-1} \qquad (pQ_T(p))' = Q'_{T-1}(p) + Q_{T-1}(p) - qQ_{T-1}(p_{T-1}) = 0.$$

Observe that $(pQ_T(p))'|_{p=p^M_{T-1}} < 0$ so the period T monopoly price should be lower than the period $T-1$ monopoly price, i.e. the price decreases as soon as

$q > 0$. The traders that accept the $T-1$ period quote make a mistake, so in this case the traders have no incentive to trade.

To summarize, in any pure or mixed strategy equilibrium there is no trading activity in the period $T-1$. There is no time discounting so the analysis of the behavior in the period $T-2$ remains the same as the analysis of the period $T-1$. The no-trading result translates inductively through period $T-2$, $T-3$, up to period 1. In any mixed or pure strategy equilibrium no transactions are concluded prior to the last period, the monopolist sets the quotes at the monopoly level or higher, equal to the monopoly level in the last period. The traders accept the monopoly quote in the last period if they make a positive profit. *Q.E.D.*

Proof of Proposition 2.3

The last period maximization problem $\max_{q>0} Q(q)(q-\overline{v})$, where $\overline{v}$ denotes the common or competitive value, can be rewritten using Lemma 2.1 to $\max_{q>0}(1-\Gamma(q))(q-\mu_\Gamma)$. We will solve the maximization problem for the normal distribution, we will use symbols $\Phi(\cdot)$ for standard normal distribution cdf and $\phi(\cdot)$ for standard normal distribution pdf. In terms of these symbols the maximization problem reads

$$\max_{q>0} \left(1-\Phi\left(\frac{q-\mu}{\sigma}\right)\right)(q-\mu).$$

Substituting $\tilde{q} = q-\mu$ we get first order conditions

$$1-\Phi\left(\frac{\tilde{q}}{\sigma}\right) - \frac{\tilde{q}}{\sigma}\phi\left(\frac{\tilde{q}}{\sigma}\right) = 0$$

The solution can be numerically estimated $\frac{q-\mu}{\sigma} \doteq 0.75179$. For $\mu = 0$ the formula reads $\frac{q}{\sigma} \doteq 0.75179$.

The welfare loss

The expected welfare loss is the loss of traders with individual valuations greater than the common value $\overline{v}$ that do not trade (those that have individual value between the common value and the monopoly level). Let p assign the difference between the quote and the common value, the problem is thus shifted to the common value 0.

$$\begin{aligned}
\frac{Exp.\ loss}{2} &= \int_0^{p^M} N(((1-\Phi(p/\sigma))-(1-\Phi(p^M/\sigma)))dp = \\
&= \int_0^{p^M} N((\Phi(p^M/\sigma))-\Phi(p/\sigma)))dp = \\
&= N\left(p^M\Phi(p^M/\sigma)-\int_0^{p^M}\Phi(p/\sigma)dp\right) = \\
&= N\left(p^M\Phi(p^M/\sigma)-\int_0^{p^M}\int_{-\infty}^{p}\frac{\phi(x/\sigma)}{\sigma}dxdp\right) = \\
&= N\left(p^M\Phi(p^M/\sigma)-\frac{1}{\sigma}\int_0^{p^M}\int_0^{p}\phi(x/\sigma)dxdp-p^M\Phi(0)\right)
\end{aligned}$$

First compute the double integral

$$\begin{aligned}
\int_0^{p^M}\int_0^{p}\phi(x/\sigma)dxdp &= \int_0^{p^M}\int_0^{\tilde{p}}\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}dxd\tilde{p} = \int_0^{p^M}\int_x^{p^M}\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}d\tilde{p}dx = \\
&= \int_0^{p^M}(p^M-x)\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}dx = \\
&= \sigma p^M(\Phi(p^M/\sigma)-\Phi(0))-\sigma\int_0^{p^M}x\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}dx = \\
&= \sigma p^M(\Phi(p^M/\sigma)-\Phi(0))-\sigma\int_0^{(p^M)^2/2}\frac{e^{-y/\sigma^2}}{\sqrt{2\pi}\sigma}dy = \\
&= \sigma\left(p^M\Phi(p^M/\sigma)-p^M\Phi(0)+\frac{\phi(p^M/\sigma)}{\sigma}-\frac{\phi(0)}{\sigma}\right)
\end{aligned}$$

After substitution to the previous formula we get

$$\begin{aligned}
Exp.loss &= 2N\left(p^M\Phi(p^M/\sigma)-p^M\Phi(p^M/\sigma)+p^M\Phi(0)-\frac{\phi(p^M/\sigma)}{\sigma}+\frac{\phi(0)}{\sigma}-\right. \\
&- \left. p^M\Phi(0)\right) = \frac{2N}{\sigma}(\phi(0)-\phi(p^M/\sigma)) \doteq .0982\frac{2N}{\sigma}
\end{aligned}$$

Proof of Proposition 2.4

Assign the first period expected order book ('demand function') $B_1(p)$. Assume that the price path $\{p_t\}_{t=1}^T$ is an equilibrium price path. We will find out which traders buy in T periods. The personal valuation of the trader indifferent between buying in period t and $t+1$ satisfies $v_i^t - p_t = (1-\delta)(v_i^t - p_{t+1})$ so the indifference value in period t is

$$v_i^t = \frac{p_t - (1-\delta)p_{t+1}}{\delta}.$$

The traders with personal valuations above this value prefer to buy, the traders under this value prefer to stay.

Observe that the $t-$period order book is equal to

$$B_t(p) = \max\left(B_1(p) - \tilde{Q}_t, 0\right),$$

where $\tilde{Q}_t$ is the amount of transactions prior to the period t. Assume that the price path satisfies $p_{t+1} \leq p_t(1-\delta)$. This assumption is convenient; if the p_{t+1} is bigger there are no transactions as well as if the equality holds so the monopolist loses on the against the discount factor. If the assumption holds the total of transactions prior to period t $\tilde{Q}_t$ is equal to the period 1 amount of traders with individual valuations higher than $v_i = \frac{p_{t-1}-(1-\delta)p_t}{\delta}$. The period t order book can be written as

$$B_t(p) = \max\left(B_1(p) - B_1\left(\frac{p_{t-1} - (1-\delta)p_t}{\delta}\right), 0\right),$$

or

$$B_t(p) = \max\left(B_k(p) - B_k\left(\frac{p_{t-1} - (1-\delta)p_t}{\delta}\right), 0\right), \qquad k < t.$$

The last period problem can be written as

$$\max_{p_T} B_T(p_T)p_T. \tag{1}$$

The solution is trivially the monopoly price (the existence and uniqueness is backed by the profit function concavity assumption).

The $T-1$ period maximization problem:

$$\max_{p_{T-1}} B_{T-1}\left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)p_{T-1}+(1-\beta)B_T(p_T;p_{T-1})p_T. \qquad (2)$$

The market maker cannot commit to any price path. Everybody assumes the market maker maximizes profit in every period, so the price p_t is given by the last period maximization problem

$$\max_{p_T}\left(B_{T-1}(p_T)-B_{T-1}\left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)\right)p_T. \qquad (3)$$

The first order conditions of maximization problems (3) and (4) give the unique equilibrium. Rewriting maximization problem (2) we get

$$\begin{aligned}\max_{p_{T-1}} B_{T-1} \quad & \left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)p_{T-1}+\\ &+(1-\beta)\left(B_{T-1}(p_T)-B_{T-1}\left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)\right)p_T\\ &s.t. \quad p_T = argmax B_T(\tilde{p}_T)\tilde{p}_T.\end{aligned}$$

Its first order condition gives the equation

$$B_{T-1}\left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)+\frac{p_{T-1}-(1-\beta)p_T}{\delta}B'_{T-1}\left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)=0$$

The fraction $\frac{p_{T-1}-(1-\delta)p_T}{\delta}$ is equal to the monopoly price with respect to the demand function B_{T-1} and the monopoly amount of traders disappear from the market as soon as $\delta=\beta$.

We can write the period t maximization problem

$$
\begin{aligned}
\max_{p_t} \quad & B_t\left(\frac{p_t-(1-\delta)p_{t+1}}{\delta}\right)p_t + \\
& +(1-\beta)\left(B_t\left(\frac{p_{t+1}-(1-\delta)p_{t+2}}{\delta}\right)-B_t\left(\frac{p_t-(1-\delta)p_{t+1}}{\delta}\right)\right)p_{t-1}+ \\
& +(1-\beta)^2\left(B_t\left(\frac{p_{t+2}-(1-\delta)p_{t+3}}{\delta}\right)-B_t\left(\frac{p_{t+1}-(1-\delta)p_{t+2}}{\delta}\right)\right)p_{t-2}+ \\
& +\ldots+ \\
& +(1-\beta)^{T-t}\left(B_t(p_T)-B_t\left(\frac{p_{T-1}-(1-\delta)p_T}{\delta}\right)\right)p_T, \\
& \quad s.t. \quad p_{t+1},\ldots,p_T \; satisfy\; respective\; maximization\; problems
\end{aligned}
$$

that gives the first order conditions

$$
B_t\left(\frac{p_t-(1-\delta)p_{t+1}}{\delta}\right)+\frac{p_t-(1-\beta)p_{t+1}}{\delta}B_t'\left(\frac{p_t-(1-\delta)p_{t+1}}{\delta}\right)=0. \quad (4)
$$

For any period $t=1,\ldots,T-1$ the fraction $\frac{p_t-(1-\delta)p_{t+1}}{\delta}$ is equal to the one shot monopoly price with respect to the residual demand, the last period price is equal to the monopoly price with respect to the residual demand, and the quantity traded is equal to the one shot monopoly quantity with respect to the residual demand as soon as $\delta=\beta$. In the case of inequality the price and quantity are changed to satisfy equation (4). *Q.E.D.*

Proof of Proposition 2.5

Assume that the traders use a mixed strategy 'buy if you can' with probability r in the first period. In the second period a new group of traders come to the market and add to the demand of those who stay there from the first round. The resulting demand function is equivalent to the situation when the demand is twice as large as the initial demand and traders use a mixed strategy 'buy if you can' with probability $r/2$. In Proposition 2.2 we have shown that $r = 0$ otherwise the one shot game monopoly price decreases and those who traded in the first period would have made a mistake. So in the first period there is no trading. By induction there are no transactions prior to the last period. *Q.E.D.*

Proof of Proposition 2.6

The proof of proposition 2.6 follows the proof of Proposition 2.4 . The idea and the line of the proof is identical, just the demand function changes.

The $t^{th}-$period indifference individual value is equal to

$$v_i^t = \frac{p_t - (1-\delta)p_{t+1}}{\delta}$$

as before. Assign the expected demand function of one trader $B(p)$ so that each period the order book increases by $\lambda B(p)$. The period t order book is equal to

$$B_t(p) = \max\left(B_{t-1}(p) - B_{t-1}\left(\frac{p_{t-1} - (1-\delta)p}{\delta}\right), 0\right) + \lambda B(p)$$

or

$$B_t(p) = \max\left((t-1)\lambda B(p) - (t-1)\lambda B\left(\frac{p_{t-1} - (1-\delta)p}{\delta}\right), 0\right) + \lambda B(p)$$

The $t^{th}-$ period maximization problem can be formulated as

$$\begin{aligned}\max_{p_t \geq 0} \quad & B_t\left(\frac{p_t - (1-\delta)p_{t+1}}{\delta}\right)p_t + \\ & +(1-\beta)\left(B_t\left(\frac{p_{t+1} - (1-\delta)p_{t+2}}{\delta}\right) + \right. \\ & \left. +\lambda B\left(\frac{p_{t+1} - (1-\delta)p_{t+2}}{\delta}\right) - B_t\left(\frac{p_t - (1-\delta)p_{t+1}}{\delta}\right)\right)p_{t+1} + \\ & +A(p_1, \ldots, p_t, \delta, \beta), \\ & \quad s.t. \quad p_{t+1}, \ldots, p_T \; satisfy \; respective \; maximization \; problems, \\ & where \quad \frac{\partial A}{\partial p_i} = 0 \quad \forall i \leq t\end{aligned}$$

This maximization problem yield F.O.C. identical to the general F.O.C. of the Proof 2.4 (the formula (4)).

$$B_t\left(\frac{p_t-(1-\delta)p_{t+1}}{\delta}\right)+\frac{p_t-(1-\beta)p_{t+1}}{\delta}B_t\left(\frac{p_t-(1-\delta)p_{t+1}}{\delta}\right)=0.$$

Seeing the problem from the viewpoint of the current demand the gradual inflow of traders does not alter the result and the conclusion is equivalent to the conclusion of Proposition 2.4. *Q.E.D.*

Proof of Proposition 2.7

The conclusion follows directly from Proposition 2.5 and the fact that the one shot game monopoly price with respect to the newcomers themselves decreases. Because the monopoly price with respect to the current demand is decreasing over the course of the day, and the traders are patient, there is no trade concluded prior to the last period.

Appendix 3
Proofs from the Section III

Proof of Proposition 3.2

The proof follows the line of the proofs 2.4 and 2.6, the result is, however, different. As before, the trading day has a last period T, the proposition is proven by a backward induction. Keep in mind that the initial period demand $B_1(p)$ relates to the distribution function of individual valuations $\Gamma(p)$ so that $B_1(p) = 1-\Gamma(p)$.

Assume that the price path $\{p_t\}_{t=1}^T$ is an equilibrium price path. We will find which traders buy in the T periods. The trader indifferent between buying in period t and $t+1$ satisfies $v_i^t - p_t = (1-\delta)v_i^t - p_{t+1}$[22] so the indifference value in period t is

$$v_i^t = \frac{p_t - p_{t+1}}{\delta}.$$

The traders above this value prefer to buy, the traders under this value prefer to stay.

Observe that the $t-$period order book is equal to

$$B_t(p) = \max\left(B_1\left(\frac{p}{(1-\delta)^{t-1}}\right) - Q_t, 0\right),$$

where Q_t is the amount of transactions prior to the period t. Assume that the price path satisfies $p_{t+1} \leq p_t(1-\delta)$. This assumption is convenient; if the p_{t+1} is bigger there are no transactions as well as if the equality holds. In this case the period t order book can be written as

$$B_t(p) = B_1\left(\frac{p}{(1-\delta)^{t-1}}\right) - B_1\left(\frac{p_t - p_{t-1}}{\delta}\frac{1}{(1-\delta)^{t-2}}\right),$$

or

$$B_t(p) = B_{t-1}\left(\frac{p}{(1-\delta)}\right) - B_{t-1}\left(\frac{p_t - p_{t-1}}{\delta}\right).$$

The last period problem can be written as

$$\max_{p_T} B_T(p_T)p_T. \tag{5}$$

The solution is trivially the monopoly price (the existence and uniqueness is backed by the profit function concavity assumption).

[22] Note that the indifference condition is similar but not identical to section 2 proofs 2.4 and 2.6 conditions.

The $T-1$ period maximization problem:

$$\max_{p_{T-1}} B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)p_{T-1} + \left(B_{T-1}\left(\frac{p_T}{1-\delta}\right) - B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)\right)p_T. \qquad (6)$$

The market maker cannot commit to any price path. Everybody assumes the market maker maximizes profit in every period, so the price p_t is given by the last period maximization problem

$$\max_{p_T} \left(B_{T-1}\left(\frac{p_T}{1-\delta}\right) - B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)\right)p_T. \qquad (7)$$

The first order conditions of maximization problems (7) and (8) give the unique equilibrium. The first order condition of the period $T-1$ maximization problem gives the condition

$$B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right) + \frac{p_{T-1}-p_T}{\delta}B'_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right) = 0 \qquad (8)$$

The fraction $\frac{p_{T-1}-p_T}{\delta}$ is equal to the monopoly price with respect to the demand function B_{T-1} and the monopoly amount of traders disappear from the market.

We can write the period t maximization problem

$$\begin{aligned}\max_{p_t} \quad & B_t\left(\frac{p_t-p_{t+1}}{\delta}\right)p_t + \left(B_t\left(\frac{p_{t+1}-p_{t+2}}{\delta(1-\delta)}\right) - B_t\left(\frac{p_t-p_{t+1}}{\delta}\right)\right) + \\ & + \left(B_t\left(\frac{p_{t+2}-p_{t+3}}{\delta(1-\delta)^2}\right) - B_t\left(\frac{p_{t+1}-p_{t+2}}{\delta(1-\delta)}\right)\right) + \ldots + \\ & + \left(B_t\left(\frac{p_T}{(1-\delta)^{T-t}}\right) - B_t\left(\frac{p_{T-1}-p_T}{\delta(1-\delta)^{T-t-1}}\right)\right),\end{aligned}$$

that gives the first order conditions

$$B_t\left(\frac{p_t-p_{t+1}}{\delta}\right) + \frac{p_t-p_{t+1}}{\delta}B'_t\left(\frac{p_t-p_{t+1}}{\delta}\right) = 0. \qquad (9)$$

For any period $t = 1, \ldots, T-1$ the fraction $\frac{p_t - p_{t+1}}{\delta}$ is equal to the one shot monopoly price with respect to the current period residual demand. The last period maximization problem (5) results in monopoly price and quantity with respect to the last period residual demand. *Q.E.D.*

Proof of Proposition 3.3

The proof is a combination of the proofs 3.2 and 2.6. Assign the expected demand function of one trader $B(p)$ so each period the order book increases by $\lambda B(p)$. Assume that the price path $\{p_t\}_{t=1}^{T}$ is an equilibrium price path. The indifference condition $v_i^t - p_t = (1-\delta)v_i^t - p_{t+1}$ leads to

$$v_i^t = \frac{p_t - p_{t+1}}{\delta}.$$

The traders above this value prefer to buy, the traders under this value prefer to stay.

The $t-$period order book is equal to

$$B_t(p) = \max\left(\sum_{i=1}^{t} \lambda B\left(\frac{p}{(1-\delta)^{t-i}}\right) - Q_t, 0\right),$$

where Q_t is the amount of trades prior to the period t. Assume that the price path satisfies $p_{t+1} \leq p_t(1-\delta)$. The assumption is backed by the fact that setting prices not satisfying the condition leads to a no-trade period and the monopolist would lose. In this case the period t order book can be written as

$$B_t(p) = B_{t-1}\left(\frac{p}{(1-\delta)}\right) + \lambda B(p) - B_{t-1}\left(\frac{p_{t-1} - p}{\delta}\right).$$

The last period problem can be written as

$$\max_{p_T} B_T(p_T)p_T, \tag{10}$$

That leads to the monopoly price.

The $T-1$ period maximization problem:

$$\max_{p_{T-1}} \quad B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)p_{T-1}+ \\ +\left(B_{T-1}\left(\frac{p_T}{1-\delta}\right)+B\left(\frac{p_T}{1-\delta}\right)-B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)\right)p_T, \\ s.t. \quad p_T\ maximizes\ (10)$$

The first order condition gives

$$B_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)+\frac{p_{T-1}-p_T}{\delta}B'_{T-1}\left(\frac{p_{T-1}-p_T}{\delta}\right)=0 \tag{11}$$

The fraction $\frac{p_{T-1}-p_T}{\delta}$ is equal to the monopoly price with respect to the demand function B_{T-1} and the monopoly amount of traders disappears from the market.

We can write the period t maximization problem

$$\max_{p_t} \quad B_t\left(\frac{p_t-p_{t+1}}{\delta}\right)p_t+ \\ +\left(B_t\left(\frac{p_{t+1}-p_{t+2}}{\delta(1-\delta)}\right)+\lambda B\left(\frac{p_{t+1}-p_{t+2}}{\delta(1-\delta)}\right)-B_t\left(\frac{p_t-p_{t+1}}{\delta}\right)\right)+ \\ +\left(B_t\left(\frac{p_{t+2}-p_{t+3}}{\delta(1-\delta)^2}\right)+\lambda B\left(\frac{p_{t+2}-p_{t+3}}{\delta(1-\delta)^2}\right)-B_t\left(\frac{p_{t+1}-p_{t+2}}{\delta(1-\delta)}\right)\right)+\ldots+ \\ +\left(B_t\left(\frac{p_T}{(1-\delta)^{T-t}}\right)+\lambda B\left(\frac{p_T}{(1-\delta)^{T-t}}\right)-B_t\left(\frac{p_{T-1}-p_T}{\delta(1-\delta)^{T-t-1}}\right)\right),$$

that gives the first order conditions

$$B_t\left(\frac{p_t-p_{t+1}}{\delta}\right)+\frac{p_t-p_{t+1}}{\delta}B'_t\left(\frac{p_t-p_{t+1}}{\delta}\right)=0. \tag{12}$$

We conclude as in the previous proof that for any period $t=1,\ldots,T-1$ the fraction $\frac{p_t-p_{t+1}}{\delta}$ is equal to one shot monopoly price with respect to the current period residual demand. In the last period the monopoly price is set and monopoly quantity traded. *Q.E.D.*

CERGE-EI
P.O.BOX 882
Politických vezòù 7
111 21 Prague 1
Czech Republic
http://www.cerge-ei.cz