Optimal Replacement Strategies in Heat Transmission Systems

Jacek A. Cukrowski
CERGE-EI, Prague, Czech Republic

February 1996

Abstract

In this paper, the issue of maintaining an optimal level of services provided by heat transmission systems working in indefinite time is analyzed. In such systems, a stable level of services can be preserved only if parts of the heat transmission pipelines are periodically replaced. It is shown that the replacement policies that maximize total discounted net benefits in indefinite time are stationary (i.e., intervals of the pipelines of the same length have to be replaced in each period). The paper presents methods for determining stationary replacement policies in homogeneous and non-homogeneous heat transmission pipelines. Moreover, it is shown that, for the purpose of analyzing stationary replacement, complex heat transmission networks can be decomposed into a set of non-homogeneous pipelines. Such a decomposition simplifies the analysis and makes the methods presented useful for investigating heat transmission networks of any size.

Abstrakt

V článku je analyzován problém optimální úrovně služeb, které poskytují systémy přenosu tepla pracující v neomezeném čase. Stabilní úroveň služeb v takových systémech může být zachována jen když se části systému přenosu tepla periodicky vyměňují. Je zde poukázáno na fakt, že výměnné postupy, které maximalizují celkové diskontované užitky v neomezeném čase, jsou stacionární (tzn., že v každé periodě je nutné vyměnit stejně dlouhou část potrubí). Článek uvádí metody pro určování postupu stacionárních výměn u homogenních a nehomogenních systémů přenosu tepla. Je zde také ukázáno, že při analýze stacionárních výměn mohou být komplexní sítě přenosu tepla dekomponovány do množiny nehomogenních potrubí. Taková dekompozice zjednodušuje analýzu a dělá ze zde uvedených metod užitečný prostředek k vyšetřování sítí přenosu tepla libovolných velikostí.

Introduction

In the broad class of engineering systems providing services of different kinds, such as lighting, heating or transportation, the main determinant of leading economic indicators is the level of performance of durable equipment. This level usually decreases with the age of the system, and, therefore, a stable flow of services can be provided only if certain parts of the equipment are replaced from time to time by new ones. Such replacements, on the one hand, reduce the number of failures, but, on the other hand, induce additional costs and therefore have to be analyzed in an economic context.

The issue of the management of durable equipment has been frequently analyzed in the economic literature. An overview of these studies has been presented, for instance, by Arrow et al. (1958), Haavelmo (1960), and Zabel (1963). Impetus for the development of this part of economic theory has come largely from practical issues such as optimal maintenance of complex electronic equipment, modern aircraft, computers, communication equipment, spacecraft, optimal replacement in the military systems, or optimal managing of the stock of machines in the production sectors (see, e.g., Jorgenson et al., 1967, or Wagner, 1975).

Spatially distributed line transmission or transport systems, such as gas and oil pipelines, energy supply lines, conveyors in manufacturing systems, power transmission lines, or heat and water pipelines, state important classes of engineering systems and have been also frequently analyzed in an economic context (see, for example, Valqui, 1978; Osiadacz, 1987; De Wolf and Smeers, 1993).

The present paper focuses on the optimal management of durable equipment in heat transmission systems which are used indefinitely. However, the methods presented can also be applied to other kinds of transmission systems (under the assumption that replacements are periodic, the time between replacements is exogenously given, and the set of conditions concerning technical characteristics is satisfied).

1. The Heat Transmission System

A heat transmission system can be defined as a set of interconnected pipelines that link spatially distributed points: a source of the heat energy with a set of receivers. The elements of this system (heat transmission pipelines) differ in their technical parameters, such as insulation or diameter, and, consequently, affect the operational reliability of the entire heating system differently.

At every moment of the system's performance, the pipeline can be in one of two extreme states: good or failed. When the pipeline is good, it performs precisely according to specification (i.e., it transmits the heat energy); when it has failed, there is a break in transmission. Failures of the pipeline transfer it from the good state to the failed. They can occur at any point in the pipeline (i.e., on an arbitrarily small length of the line, an infinite number of failures can occur). The only action which transfers the pipeline from a failed state to a good state is a repair which consists of the replacement of a small piece of failed pipeline just at the point where the failure occurred. It means that each repair renews only a very small section of the pipeline. This section is so short in comparison with the length of the pipeline that the repair does not change the reliability characteristics of the considered pipeline as a whole (Stepien, 1991a), as is usually assumed. Hence, for the purpose of reliability analysis, the transmission line can be represented as an infinite sequence of pieces (renewable elements) with zero length (Stepien, 1991a). The reliability characteristics of every separate piece (i.e., of every renewable element) cannot be measured, but the reliability indicators of the pipeline with respect to the standard interval can be estimated empirically (Stepien, 1991a).

The main reliability characteristics of the lifetime of the heat pipeline is the pipeline's failure rate at a given moment t ($\lambda(t)$), defined as the probability that the pipeline will fail in the interval (t,t+dt) under the condition that it survived until moment t:

$$\lambda(t) = \frac{f(t)}{1 - F(t)} , \quad t \geq 0 \tag{1}$$

where F(t) is a cumulative probability function of the lifetime of the heat pipeline, and f(t)=dF(t)/dt is its probability density function.

It has been shown by Stepien (1991a, 1991b) that the lifetime of heat pipelines is described by a random variable with a Weibull distribution. Consequently, the failure rate of the standard interval of the pipeline is specified by

$$\lambda(t) = \frac{1}{\gamma}\alpha\left(\frac{1}{\gamma}t\right)^{\alpha-1} , \quad t \geq 0 , \tag{2}$$

where $\alpha > 0$, $\gamma > 0$ (see Figure 1). The empirically determined values of parameters α and γ, for some types of heat pipelines coming from heat transmission systems in Poland (Stepien 1991a, 1991b), are presented in Table 1.

Figure 1
Shapes of the Functions of Failure Rates in Weibull Distribution for
A: $0 < \alpha < 1$, B: $\alpha = 1$, C: $1 < \alpha < 2$, D: $\alpha = 2$, E: $\alpha > 2$

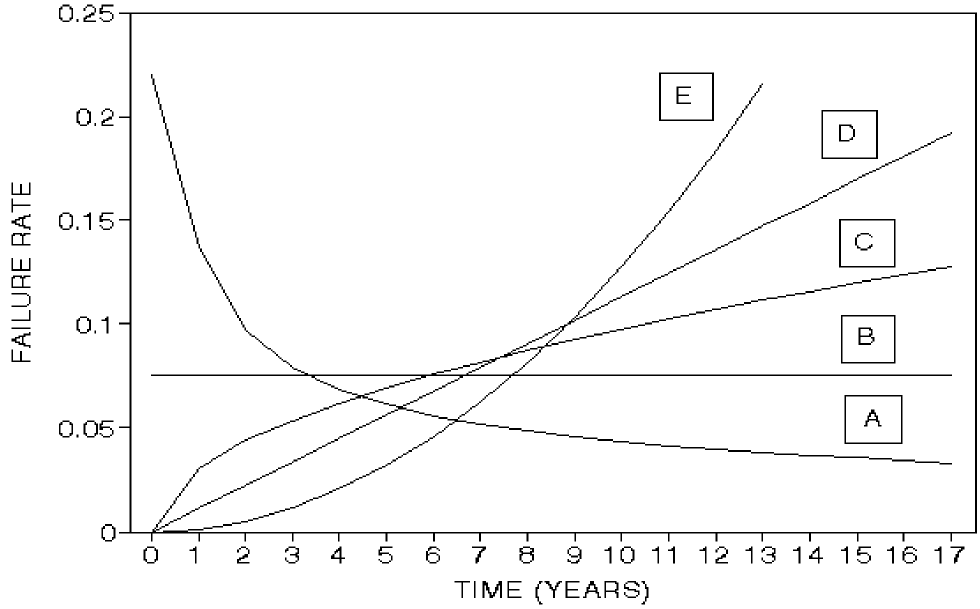


Table 1
Values of parameters α and γ of Weibull distributions describing the lifetime of commonly used types of pipelines (for 100m)

No.	Heat insulation	Diameter (mm)	α	γ
1.	Asbestos cement	700	3.9	28.5
2.	Asbestos cement	150	5.2	25.0
3.	Asbestos cement	80	9.0	13.0
4.	Foamed concrete	125	9.0	25.0
5.	Foamed concrete	50	9.0	13.0
6.	Glass wool	700	5.2	25.0
7.	Glass wool	150	9.0	13.0
8.	Fill mass	150	9.0	13.0
9.	Fill mass	50	4.7	13.5

One can note that, for each particular type of pipeline, the value of parameter α is greater than 2. This corresponds to a very intensive ageing process characterized by curve E in Figure 1.

After each failure, the pipeline is repaired by replacing a piece of it. The durations of these repairs are stochastic and depend on the technical characteristics of the pipeline repaired (mostly on its diameter), as well as on the organizational properties of the repair teams. The empirical analysis shows that the durations of these repairs can be described by a negative exponential distribution (see Stepien, 1991b).

To consider the performance of the heating system in discrete time, let us denote the failure rate of a certain standard interval (e.g., 100m) of i periods old pipeline as λ_i (i=0,1,2,...) and assume that the duration of a single period equals one year. The failure rate λ_i increases with the number of periods (i=0,1,2...) corresponding to the ageing process (see curve E in Figure 1). An increase in the number of failures decreases the level of services provided. A natural way to improve the reliability of the system (i.e., to decrease the failure rate) is to use breaks between heating seasons to replace the oldest parts of heat transmission pipelines. The method for determining the optimal replacements is discussed below.

2. Optimal Replacement Policy for the Homogeneous Pipeline

Assuming that the replacement is carried out during breaks between heating seasons, the objective of the replacement policy is to find a sequence $(l_0,l_1,l_2,...)$ of intervals of the homogeneous pipeline that have to be replaced in order to maximize the total discounted net benefits over the life of the system analyzed.

Taking into account that the lifetime of the transmission system is unlimited, the optimal policy can be derived from the following infinite-horizon, discounted, dynamic programming problem (it is assumed that the preventive replacement is done before the heating season):

$$\underset{,l_1,l_2\cdots}{\text{nax}} \sum_{t=0}^{\infty} \beta^t[u_t(\Lambda_t,l_t) - c_t \; l_t] \qquad (3)$$

where the maximization is subject to

$$\Lambda_{t+1} = g(\Lambda_t,l_t)\,, \quad \text{with } \Lambda_0 = \lambda_0\,; \qquad (4)$$

and

l_t is a control variable (the length of the pipeline replaced in period t,

$t=0,1,2,...$),

Λ_t is a state variable (the failure rate at the beginning of period t, $t=0,1,2,...$),

$u_t(\Lambda_t,l_t)$ is the net benefit (the gross benefit derived from operating the pipeline less any operating costs) in period t (t = 0,1,...) when the replacement $l_t \in [0,L]$ (L is the total length of the pipeline analyzed) is done in this period and the state of the pipeline at the beginning of the period is characterized by Λ_t;

c_t is the cost of replacing the unit of the pipeline (1m);

β is a discount factor ($\beta \in (0,1)$).

Assuming that all the parameters are stationary over time, the optimal solution to an infinite-horizon, discounted, dynamic programming problem is time-invariant (see, for example, Sargent, 1987, Chapter 1). Thus, in the problem considered, the optimal replacement policy is stationary (i.e., $l_0^*=l_1^*=l_2^*=...=l^*$).

To find l^*, consider a homogeneous pipeline with total length L, in which interval l is replaced each period. Assume first that the length of the interval replaced (l) belongs to the interval (L/2,L]. Thus, each period the pipeline considered consists of an interval (l) of new pipeline (characterized by failure rate λ_0) and an interval (L − l) of one-year-old pipeline (characterized by failure rate λ_1). Consequently, the failure rate of 1m of such a pipeline can be determined as

$$= 1-(1-\frac{l}{100}\lambda_0)(1-\frac{(L-l)}{100}\lambda_1) = \frac{1}{100}[l\lambda_0+(L-l)\lambda_1-\frac{(L-l)l}{100} \qquad (5)$$

Assuming that λ_0 and λ_1 are small numbers, the last term in the expression above can be neglected, and, finally, the dependence of the failure rate (Λ) on the length l (when $l\in(l/2,L]$) of the stationary replacement can be specified as

$$.(l) = \frac{L}{100}[\frac{l}{L}\lambda_0 + (1 - \frac{l}{L})\lambda_1] , \quad if\ L \geq l > \frac{L}{2} \qquad (6)$$

Similarly,

$$l) = \frac{L}{100}[\frac{l}{L}\lambda_0 + \frac{l}{L}\lambda_1 + (1 - \frac{2l}{L})\lambda_2] , \quad if\ \frac{L}{2} \geq l > \qquad (7)$$

$$) = \frac{L}{100}[\frac{l}{L}\lambda_0 + \frac{l}{L}\lambda_1 + \frac{l}{L}\lambda_2 + (1 - \frac{3l}{L})\lambda_3] , \quad if\ \frac{L}{3} \geq l > \qquad (8)$$

etc. Succinctly, the expressions above can be represented as

$$l) = \frac{L}{100}[\frac{l}{L}\sum_{i=1}^{n} \lambda_{i-1} + (1 - \frac{nl}{L})\lambda_n] , \quad if\ \frac{L}{n} \geq l > \frac{L}{n +} \qquad (9)$$

It turns out that the failure rate $\Lambda(l)$ is a decreasing function of the stationary replacement (l) on each interval $(L/(n+1),L/n]$, where $n=1,2,\ldots$. For each such interval (specified by n) the failure rate (Λ_n) can be represented as a linear function of the length of the stationary replacement l ($l \in (L/(n+1),L/n]$):

$$\Lambda_n(l) = a_n + b_n\, l\,, \tag{10}$$

where a_n ($a_n > 0$) and b_n ($b_n < 0$) are constants (for each fixed value of n) defined as follows:

$$a_n = \frac{L\lambda_n}{100}\,, \quad b_n = \frac{1}{100}\Big(\sum_{i=1}^{n} \lambda_{i-1} - n\lambda_n\Big)\,. \tag{11}$$

This implies that, in a steady state, the failure rate (i.e., the state variable) is fully determined by a stationary choice of l, i.e., $\Lambda(l) = \Lambda_n(l)$, if $l \in (L/(n+1),L/n]$. Consequently, in order to find the optimal length of the replacement, it is enough to consider the following optimization problem:

$$\underset{l}{Max}\ \{\ u(\Lambda(l),l) - cl\ \}\,. \tag{12}$$

where $u(\Lambda(l),l)$ is the net benefit generated in each period when a preventive replacement (l) is done during breaks between heating seasons, and $c=c_0=c_1=\ldots$ denotes the cost of replacing a unit of the pipeline.

Assuming that the total benefit generated by a heating system during one period when no failures occur (B_t) and the average cost of a single repair (k) are constant in each period, the net benefit generated in each period, $u(\Lambda(l),l)$, is determined as follows:

$$u(\Lambda(l),l) = u(\Lambda(l)) = B\ P(\Lambda(l)) - k\Lambda(l)\,, \tag{13}$$

where

B is the total (gross) benefit generated by a heating system during a single period when no failures occur;

$P(\Lambda(l))$ is the probability that the system works without breaks during the period considered, when interval l of the pipeline is replaced in each period;

$\Lambda(l)$ is the failure rate of the pipeline when interval l is replaced in each period ($\Lambda(l)$: $R_+ \to R_+$ such that $d\Lambda(l)/dl < 0$ and $d^2\Lambda(l)/dl^2 > 0$);

k is the average cost of a single repair.

Finally, the objective of the preventive replacement in the case of a homogeneous pipeline can be specified as

$$\underset{l}{Max}\ \{\ B\ P(\Lambda(l))\ -\ k\ \Lambda(l)\ -\ c\ l\ \} \tag{14}$$

The probability P(Λ(l)) that the system under study works without failures, if interval l is replaced in each period, is specified as follows:

$$P(\Lambda(l))\ =\ \frac{t_l(\Lambda(l))}{t_l(\Lambda(l))\ +\ t_r}\ =\ \frac{\mu}{\mu\ +\ \Lambda(l)} \tag{15}$$

where $t_l(\Lambda(l))$ is the average time in which the pipeline transmits the heat energy ($t_l=1/\Lambda(l)$), t_r is the average time of a single repair, and $\mu=1/t_r$ denotes the number of repairs that can be done in a unit of time.

Taking into account that

$$\Lambda(l) = \Lambda_n(l) = a_n + b_n l\ ,\quad \textit{if}\ \ l\in(L/(n+1),L/n]\ , \tag{16}$$

the objective function, when the length of the replacement $l\in(L/(n+1),L/n]$ (for each fixed value of n, n=1,2,...), can be written as

$$\underset{(\frac{L}{n+1},\frac{L}{n}]}{Max}\ \left\{\ \frac{B\mu}{\mu\ +\ a_n\ +\ b_n l}\ -\ (a_n\ +\ b_n l)k\ -\ cl\ \right\} \tag{17}$$

The first derivative of the maximized function with respect to l ($l\in(L/(n+1),L/n]$) equals zero, if

$$l = \frac{-(a_n\ +\ \mu)\ \pm\ \sqrt{\dfrac{-B\ \mu\ b_n}{b_n k\ +\ c}}}{b_n}\quad \text{and}\quad \frac{-B\ \mu\ b_n}{b_n k\ +\ c}\ \geq\ 0\ . \tag{18}$$

The expression under the root is non-negative if

$$b_n\ >\ -\frac{c}{k}\ , \tag{19}$$

i.e., if $b_n\in(-c/k,0)$.

The second derivative of the objective function equals

$$\frac{2\ b_n\ B\ \mu}{(\mu\ +\ a_n\ +\ b_n\ l)^3}\ , \tag{20}$$

and it is negative if

$$l > -\frac{a_n + \mu}{b_n} \,. \tag{21}$$

This implies that an interior maximum (in the interval (L/(n+1),L/n], n=1,2,...) exists only if

$$l' = \frac{-(a_n + \mu) - \sqrt{\frac{-B\,\mu\,b_n}{b_n k + c}}}{b_n} \in \left(\frac{L}{n+1}, \frac{L}{n}\right] \tag{22}$$

otherwise the maximum in this interval is not interior (b_n is constant in the interval (L/(n+1), L/n], n=1,2,..., and, consequently, in each interval specified by n, at most one interior maximum exists).

The objective function increases in the interval (L/(n+1),L/n] if its first derivative with respect to l is positive for all l∈(L/(n+1),L/n], i.e., if

$$-\frac{B\,\mu\,b_n}{(\mu + a_n + b_n l)^2} - (b_n k + c) > 0 \tag{23}$$

The ratio $B\mu b_n/(\mu+a_n+b_n l)^2$ is always negative; therefore, the inequality above is undoubtedly satisfied if $(b_n k+c)$ is also negative, i.e., if

$$b_n < -\frac{c}{k} \,. \tag{24}$$

The values of coefficients b_n (n = 1,2,...,) are negative and decrease with n. Therefore, for sure, there exists such $n=n^*$ starting from which b_n is smaller than -c/k (c and k are constant), and, consequently, in all intervals specified by $n \geq n^*$, the objective function is strictly increasing.

This implies that, in order to find the maximum of the objective function considered, it is not necessary to analyze more than n^* intervals, taking into account the interior maximum (if it exists) or the right boundary of the interval analyzed (note that $\Lambda(L/n) = a_n+b_n L/n$ increases with n, i.e., $\Lambda(L/(n+1)) > \Lambda(L/n)$, for n=1,2,...), and to choose the length of the replacement which maximizes the value of the objective function.

Formally, it can be formulated as the following problem of discrete optimization:

$$\max_l \left\{\frac{B\,\mu}{\mu + a_n + b_n l} - (a_n + b_n l)\,k - cl \right. \tag{25}$$

s.t. $l \in \{l_1, l_2, ..., l_{n^*}\}$

where

n^* is the smallest number of the interval in which the value of coefficient b_{n^*} is smaller than -c/k,

l_n ($n=1,2,..,n^*$) are determined as

$$\frac{+\ \mu) - \sqrt{\frac{-B\ \mu\ b_n}{b_n k + c}}}{b_n}\ ,\quad if\ \ b_n > -\frac{c}{k}\ \ and\ \ l_n \in (- \qquad (26)$$

or

$$l_n = \frac{L}{n}\ ,\quad \text{otherwise.} \qquad (27)$$

Note that l_n ($n = 1,2,...,n^*$) are not necessarily the optimal lengths of the replacements in the intervals considered. Nevertheless, the optimal length of the replacement belongs to the set $\{l_1, l_2, ..., l_{n^*}\}$.

The running time of the algorithm for determining the optimal replacement depends on the number of intervals analyzed (n^*). Taking into account that the failure rate is determined by (2), the inequality (24) can be represented as

$$b_n = \frac{1}{100}\ \frac{\alpha}{\gamma^\alpha}\ [\sum_{i=1}^{n} (i - 1)^{\alpha-1} - n^\alpha] < -\frac{c}{k}\ . \qquad (28)$$

Note that,

$$\sum_{i=1}^{n} (i - 1)^{\alpha-1} < \frac{n^\alpha}{2}\ , \qquad (29)$$

for all $\alpha > 2$. Thus, the inequality (28) is always satisfied if

$$\frac{n^\alpha}{2} > \frac{100}{\alpha}\frac{c}{k}\gamma^\alpha\ , \qquad (30)$$

and, consequently, if

$$n > (\frac{100}{\alpha}\frac{c}{k})^{\frac{1}{\alpha}}\ \gamma\ . \qquad (31)$$

The smallest integer value, $\bar{n}$, such that $\bar{n} > n$, estimates from above the number of intervals that have to be analyzed, n^* (i.e., n^* is always smaller or at most equal to $\bar{n}$).

As an example, consider L = 2600m of the homogeneous pipeline with the diameter 700mm and asbestos cement insulation (α=3.9, γ=28.5). Suppose that the total benefit generated by a heating system during a single period when no failures occur (B) equals 10^5 dollars, the average cost of a single repair (k) is 10^3 dollars, and the cost of replacing the unit of pipeline (c) equals 25 dollars. Assume also that the average time of a single repair, t_r, equals 1 day, i.e., 1/365 part of the year ($\mu = 1/t_r = 365$). The optimal replacement policy can be derived from the analysis of no more than n^*=28 intervals: (L/(n+1),L/n], where n=1,2,...,28 (the smallest integer number satisfying inequality (31): $\bar{n}$ = 31). The value of the optimal replacement (l^*) equals L/26 (100m). The corresponding value of the objective function equals 96,675.31 dollars.

3. The Non-Homogeneous Pipeline

In real heat transmission systems, transmission lines connecting the source with the receivers usually contain not one but several homogeneous pipelines. In such systems, each homogeneous pipeline works until its own failure, but every failure of another pipeline placed between it and the source also interrupts its work. This means that the pipelines which are placed closer to the source influence the work of the pipelines placed farther from it but not vice versa.

Analogously, as in the case of homogeneous pipelines, the optimal replacement policy for the transmission line with J homogeneous intervals can be derived from the following optimization problem:

$$\max_{l_{j,t},\ j=1,2,\dots,J,\ t=0,1,\dots} \sum_{t=0}^{\infty} \beta^t [u_t(\Lambda_{1,t},\dots,\Lambda_{J,t},l_{1,t},\dots,l_{J,t}) - \sum_{j=1}^{J} c_{j,t}\ \cdots \tag{32}$$

where the maximization is subject to

$$\Lambda_{1,t+1}=g(\Lambda_{1,t},l_{1,t})\ , \tag{33}$$

$$\Lambda_{2,t+1}=g(\Lambda_{2,t},l_{2,t})\ , \tag{34}$$

....

$$\Lambda_{J,t+1}=g(\Lambda_{J,t},l_{J,t})\ , \tag{35}$$

with $\Lambda_{1,0}=\lambda_{1,0}$, $\Lambda_{2,0}=\lambda_{2,0}$, ..., $\Lambda_{J,0}=\lambda_{J,0}$; (36)

$l_{1,t},l_{2,t},\dots,l_{J,t}$ are the control variables (the lengths of the pipelines replaced in period t, t=0,1,2,...),

$\Lambda_{1,t},\Lambda_{2,t},\dots,\Lambda_{J,t}$ are the state variables (the failure rates at the beginning of period t, t=0,1,2,...),

$u_t(\Lambda_{1,t},\dots,\Lambda_{1,t},l_{1,t},\dots,l_{J,t})$ is the net benefit in period t (t = 0,1,...), when replacements $l_j\in[0,L_j]$ (L_j is the total length of the pipeline j, j=1,2,...,J) are done in this period, and the states of the pipelines at the beginning of the period are characterized by $\Lambda_{j,t}$ (j=1,2,...,J):

(37)

B_t is the total return generated by the system in period t when no failures occur,

$P_t(\Lambda_{1,t},...,\Lambda_{J,t},l_{1,t},...,l_{J,t})$ is the probability that no failures occur during period t,

$k_{j,t}$ is the average cost of a single repair of pipeline j in period t;

$c_{j,t}$ is the cost of replacing a unit of pipeline j, in period t (t=0,1,2,...);

β is a discount factor ($\beta\in(0,1)$).

If all the parameters are stationary over time, the optimization problem above is a typical infinite-horizon dynamic programming problem, and, consequently, the solution to it is stationary, i.e., $l_{1,0}=l_{1,1}=l_{1,2}=...=l_1^*,..., l_{J,0}=l_{J,1}=l_{J,2}=...=l_J^*$.

Similar to the case of a homogeneous pipeline, if a stationary replacement policy is applied, then the state variables are fully determined by the values of stationary replacements $l_1,l_2,...,l_J$. Therefore, assuming that the values of B_i, k_i, c_i are constant over time, the optimization problem can be formulated as follows:

$$\underset{\substack{l_j\in(0,L_j],\\ j=1,2,...,J}}{Max} \{ B\; P(\Lambda_1(l_1),\Lambda_2(l_2),...,\Lambda_J(l_J)) - \sum_{j=1}^{J} \Lambda_j(l_j)k_j - \sum_{j=1}^{J} c_j \tag{38}$$

where

$$\frac{L_j}{100}[\frac{l_j}{L_j}\sum_{i=1}^{n_j}\lambda_{j,i-1} + (1-\frac{n_j\, l_j}{L_j})\,\lambda_{j,n_j}] , \quad if\ \frac{L_j}{n_j} \geq l_j > \tag{39}$$

(n_j=1,2,...);

$\lambda_{j,i}$ is the failure rate of pipeline j (j=1,2,...,J) in period i (i=1,2,...).

Similar to the case of a single homogeneous pipeline, one can show that there exist n_j^* (j = 1,2,...,J) such that the objective function increases with l_j if $l_j\in(L_j/(n_j + 1),L_j/n_j]$, for all $n_j > n_j^*$ (j = 1,2,...,J), and, consequently, that the maximum of the objective function can be determined (analytically or using numerical methods) on a finite number of the intervals.

The crucial point of the analysis is the determination of the probability $P(\Lambda_1(l_1),\Lambda_2(l_2),...,\Lambda_J(l_J))$ of the working state of the system under study. Taking into account that

- a stationary replacement policy makes the failure rate constant, i.e., it converts the Weibull distribution of the lifetime of each homogeneous pipeline into a negative exponential distribution, and
- the repair time is a random variable with negative exponential distribution,

this probability can be determined using the Markov chains technique (see Kleinrock, 1975, for details). The corresponding method for determining the probability of the "good" state of the non-homogeneous pipeline as the function of the stationary replacements $l_1, l_2, ..., l_J$ is presented below.

If the indicator of the state of the homogeneous pipeline j ($j = 1,2,...,J$) is denoted as

$$a_j = \begin{cases} 1 \text{ , if the homogeneous pipeline j is repaired,} \\ 0 \text{ , otherwise,} \end{cases} \tag{40}$$

then the states of the heat transmission system with J homogeneous pipelines can be described by the vectors $S_m = (a_1, a_2, ..., a_J)$, $a_j \in \{0,1\}$ ($j = 1,2,...,J$). Subscripts m ($m = 1,...,2^J$) which identify the states (S_m) are specified as $m = N_b + 1$, where N_b is a decimal representation of the binary number specified by the sequence $a_1, a_2, ..., a_J$ of elements of the corresponding vector. Each state (S_m, $m = 1,2,...,2^J$) can be represented as a point in a J-dimensional Euclidean space with coordinates $(a_1, a_2, ..., a_J)$. Thus, transitions from the one state to another are possible only if these states differ by only one coordinate (i.e., if the Euclidean distance between these states equals one). Formally, the state-transition-rate diagram of the Markov chain which corresponds to the non-homogeneous pipeline with J homogeneous intervals is specified by the directed graph:

$$\mathbf{G} = (\mathbf{S}, \mathbf{D}, u) \text{ ,} \tag{41}$$

where

S is a set of vertices which is equal to the set of states
$\mathbf{S} = \{S_1, S_2, ..., S_{2^J}\}$,

D is a set of directed arcs ($\mathbf{D} \subset \mathbf{S} \times \mathbf{S}$), such that the arc (S_m, S_z) belongs to **D** in the two following cases:

(1) if the vectors describing states S_m and S_z differ only by the value of one coordinate (say y, $1 \leq y \leq J$), and $m > z$;

(2) if vectors describing states S_m and S_z differ only by the value of one coordinate (say y, $1 \leq y \leq J$), the first y coordinates of the vector corresponding to state S_m equal zero (i.e., $a_1 = a_2 = ... = a_y = 0$), and $m < z$.

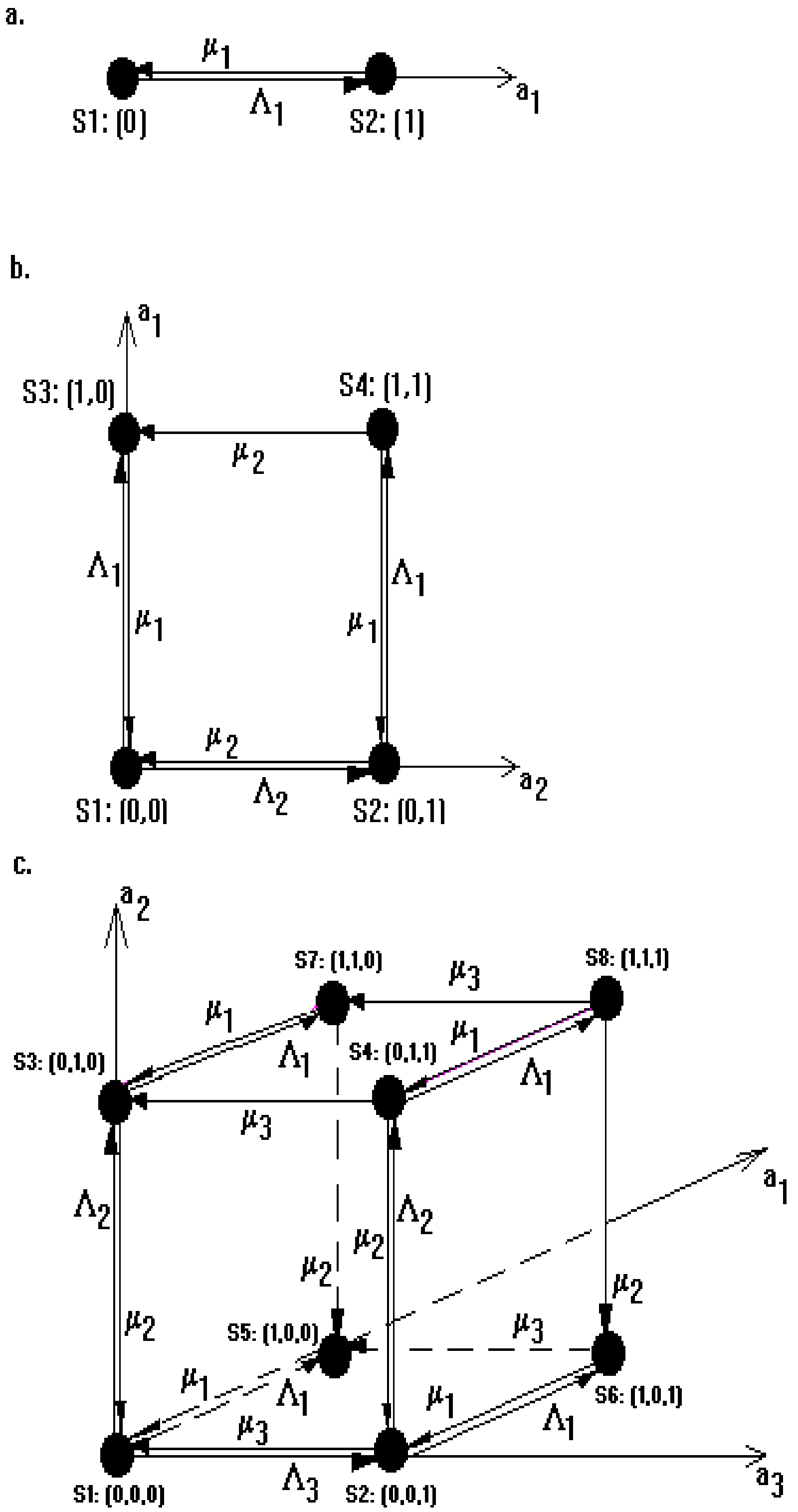
u is a transition rate function that assigns the following numbers to the arcs

(S_m,S_z) $((S_m,S_z) \in \mathbf{D})$ of graph **G**:

$$u(S_m,S_z) = \begin{cases} \mu_y, \text{ if } (S_m,S_z) \text{ is a subject to the case 1,} \\ \Lambda_y, \text{ if } (S_m,S_z) \text{ is a subject to the case 2.} \end{cases} \quad (42)$$

The state-transition-rate diagrams for pipelines with J = 1, J = 2, and J = 3 homogeneous lines are presented in Figure 2.

Figure 2
State-Transition-Rate Diagram for a Non-Homogeneous Pipeline
with (a) J=1, (b) J=2, (c) J=3 Homogeneous Intervals



The vector of stationary probabilities $\pi = [\pi_1, \pi_2, ..., \pi_{2^J}]$, where π_m is the stationary probability that the system considered is in state m (m = 1,2,...,2^J), can be calculated from the following system of linear equations:

$$\mathbf{Q}\,\pi = \mathbf{0} \qquad (43)$$

$$\sum_{m=1}^{2^J} \pi_m = 1 \ , \qquad (44)$$

where matrix **Q** is the infinitesimal generator of the Markov process:

$$\mathbf{Q} = \left[\, q_{m,z} \,\right], \quad m,z = 1,2,...,2^J \ ; \qquad (45)$$

where elements outside the main diagonal $q_{m,z}$ ($m \neq z$) are rates of transition from state S_m to state S_z, i.e., $q_{m,z} = u(S_m, S_z)$, ($m,z = 1,2,...,2^J$, $m \neq z$), while the elements on the main diagonal $q_{m,m}$ make the sum of elements of each row equal to zero, i.e.,

$$q_{m,m} = -\sum_{z=1,...,2^J,\ z \neq m} u(S_m, S_z) \ . \qquad (46)$$

The equations specified by the first expression are linearly dependent, thus one of them should be neglected.

The probability of state S_1:(0,0,...,0), i.e., the probability that the non-homogeneous pipeline transmits the heat energy, is determined as

$$\pi_1 = \det(\mathbf{H}^1)/\det(\mathbf{H}) \ . \qquad (47)$$

Matrices **H** and $\mathbf{H}^1$ are specified as follows:

$$\mathbf{H} = \left[\, h_{m,z} \,\right], \quad m,z = 1,2,...,2^J \ ; \qquad (48)$$

where

$$h_{m,z} = \begin{cases} 1 \ , & \text{if } m = 2^J, \\ q_{m,z} \ , & \text{otherwise,} \end{cases} \qquad (49)$$

and

$$\mathbf{H}^1 = \left[\, h^1_{m,z} \,\right], \quad m,z = 1,2,...,2^J \ ; \qquad (50)$$

where

$$h^{1}_{m,z} = \begin{cases} 0\,, & \text{if } z = 1 \text{ and } m < 2^{J}\,, \\ h_{m,z}\,, & \text{otherwise.} \end{cases} \tag{51}$$

The expression (47) represents the probability that the non-homogeneous pipeline works without any breaks during the period considered, as a function of the failure rates of the homogeneous pipelines, and, consequently, as a function of the stationary replacements l_j, $j=1,2,...J$ (see Cukrowski, 1993, for details). Note that, in the real world, even in large heat transmission networks, the number of homogeneous intervals (J) in non-homogeneous pipelines is usually small (see Section 5), and, consequently, determination of the probability π_1 is relatively simple.

Knowing the relationship between the lengths of the stationary replacements and the probability that the non-homogeneous pipeline works without any brakes during the period considered:

$$P(\Lambda_1(l_1),\Lambda_2(l_2),...,\Lambda_J(l_J)) = \pi_1 = \det(\mathbf{H}^1)/\det(\mathbf{H})\,, \tag{52}$$

the lengths of the optimal stationary replacements can be determined by finding the solution to the maximization problem specified at the beginning of this section. However, for complex pipelines (i.e., for large number of homogeneous intervals), a numerical analysis has to be applied (see Cukrowski, 1993, for an example).

4. Heat Transmission Networks

The heat transmission pipelines considered in the previous sections are usually included into networks where many receivers are connected with only one source of heat energy. A simple case of a heat transmission network (three receivers and five intervals of homogeneous transmission lines) is presented in Figure 3.

In such a network every receiver is connected with the source by a set of homogeneous pipelines Ξ_q (q=1,2,...,Q, where Q denotes the number of receivers).

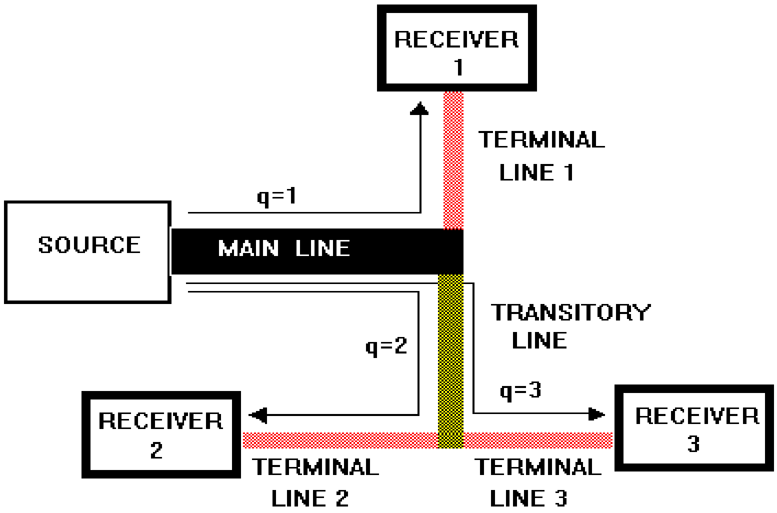
An optimal stationary replacement policy for the heat transmission network can be derived from the following optimization problem:

$$\underset{\substack{L_J],\\ ,...,\ J}}{ax} \ \{ \sum_{q=1}^{Q} B_q P_q(\Lambda_1(l_1),\Lambda_2(l_2),...,\Lambda_J(l_J)) - \sum_{j=1}^{J} \Lambda_j(l_j)k_j - \sum_{j=1}^{J} c_j \tag{53}$$

$$\text{where } \ l_j) = \frac{L_j}{100}[\frac{l_j}{L_j}\sum_{i=1}^{n_j} \lambda_{j,i-1} + (1 - \frac{n_j l_j}{L_j})\lambda_{j,n_j}]\ , \quad if\ \frac{L_j}{n_j} \geq l_j > \frac{}{n_j} \tag{54}$$

$n_j = 1,2,\dots, \quad j = 1,2,\dots, J;$

Figure 3
A Simple Structure of a Heat Transmission Network



B_q denotes the return generated by connection q in a single period when no failures occur;

$P_q(\Lambda_1(l_1),\Lambda_2(l_2),\dots,\Lambda_J(l_J))$ is the probability that connection q works during the period considered, if the intervals $l_1,l_2,\dots,l_J$ are replaced in each period;

$\Lambda_j(l_j)$ denotes the failure rate of pipeline j, if the stationary replacement equals l_j,

k_j is the average cost of a single repair of pipeline j;

c_j is the average cost of replacement of one unit of pipeline j;

L_j is the length of pipeline j;

j specifies the homogeneous pipeline (j = 1,2,...,J, J is the number of homogeneous pipelines in the network considered).

The probabilities P_q (q = 1,2,...,Q), as functions of the lengths of the replaced intervals, can be derived from the Markovian analysis of the transmission network. However, the analysis of the network with J intervals of homogeneous pipelines leads to a Markov chain with 2^J states, and, consequently, to a system of 2^J linear equations. The solution to this system gives expressions for the probabilities P_q

(q=1,2,...,Q), but, in general, if the number of intervals of homogeneous pipelines in the network is large, such an analysis faces serious computational problems. These problems can be avoided if every connection source-receiver specified in the network is analyzed separately (typically each connection contains only a few homogeneous intervals). A decomposition of the network is possible, because the probability (P_q) that receiver q (q=1,2,...,Q) works during the period considered depends only upon the failure rates of the homogeneous pipelines that connect receiver q with the source (i.e., upon the pipelines that belong to the set Ξ_q). This implies that the probabilities P_q (q = 1,2,...,Q), as functions of the lengths of intervals of homogeneous pipelines replaced in each period, can be derived from the analysis of separated non-homogeneous pipelines specified by the sets Ξ_q (as in Section 4), and the lengths of optimal replacements can be computed from the analysis of the above optimization problem (a numerical example of this analysis is presented in Cukrowski, 1993). It turns out that the complexity of analysis is determined not by the number of homogeneous pipelines in the network, but by the maximum number of pipelines in a single connection source-receiver.

Conclusion

A stable level of services provided by heat transmission systems can be maintained only if some parts of durable equipment (heat transmissions pipelines) are periodically replaced. The paper shows that the replacement policy which maximizes total discounted net benefits in the case when the system works indefinitely is stationary, i.e., that intervals of pipelines of the same length have to be replaced in each period.

In the simplest heat transmission system (a single homogeneous pipeline), the optimal length of the interval replaced can be determined through a simple problem of discrete optimization. Optimal replacement policies in complex heat transmission systems can be easily computed numerically under the condition that the probabilities of the working state of the non-homogeneous pipelines connecting the source with the receivers are specified as functions of the lengths of the intervals replaced. Stationary replacement and, consequently, an exponentially distributed time between failures (when the replacement policy is applied) allow us to determine these probabilities with the help of the Markov chain technique.

Optimal replacement policies for heat transmission networks can also be found numerically, based on expressions for probabilities of the network's states. These probabilities can be derived from the Markov chain of the whole network.

However, the complexity of the analysis significantly decreases if the network considered is decomposed into a set of connections (non-homogeneous pipelines linking the source with the receivers), and the expressions for the corresponding probabilities are derived in a set of separate models. Such a decomposition makes the methods presented in the paper useful for the analysis of replacement policies in heat transmission networks of any size.

REFERENCES

Arrow, K. J., S. Karlin, and H. Scarf, eds. 1958. *Studies in the Mathematical Theory of Inventory and Production*. Stanford, CA: Stanford University Press.

Cukrowski, J. A. 1993. "Optimal Managing of Durable Equipment in Heat Energy Transport Systems," TAEG, Vienna University of Economics & Business Administration, WSG-Discussion Papers, No. 33.

De Wolf, D. and Y. Smeers. 1993. "Optimal Dimensioning of Pipe Networks with Application to Gas Transmission Networks," CORE Discussion Papers, No. 9315.

Haavelmo, T. 1960. *A Study in the Theory of Investment*. Chicago: University of Chicago Press.

Jorgenson, D. W., J. J. McCall, and R. Radner. 1967. *Optimal Replacement Policy*. Amsterdam: North-Holland Publishing Company.

Kleinrock, L. 1975. *Queuing Systems*, *Vol. 1: Theory*. New York: John Wiley.

Osiadacz, A. J. 1987. *Simulation and Analysis of Gas Networks*. London: E. and F. N. Spon.

Sargent, T. J. 1987. *Dynamic Macroeconomic Theory*. Cambridge, MA: Harvard University Press.

Shamir, U. 1974. "Optimal Design of Operation of Water Distribution Systems," *Water Resources Research* 10,1: 27–36.

Stepien, J. C., W. Smaga, and M. Strzelecki. 1985. "Wpływ sezonowości na zawodność sieci cieplnej," *Ciepłownictwo.Ogrzewnictwo.Wentylacja* 2: 35–41.

Stepien, J. C. 1991a. "Ocena funkcji niezawodności sieci cieplnych," *Ciepłownictwo.Ogrzewnictwo.Wentylacja* 9: 199–201.

———. 1991b. "Modele niezawodności i czasu trwania awarii sieci cieplnych," *Ciepłownictwo.Ogrzewnictwo.Wentylacja* 10: 222–25.

Valqui Vidal, R. V. 1978. "Design of a Pipeline Route," *Mathematical Programming Study* 9: 95–105.

Wagner, H. M. 1975. *Principles of Operations Research with Applications to Managerial Decisions*. Englewood Cliffs, NJ: Prentice-Hall, Inc.

Zabel, E. 1963. "Efficient Accumulation of Capital for the Firm," *Econometrica* 31: 131–50.