STAGES OF GROWTH IN ECONOMIC DEVELOPMENT*

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Abstract

The paper analyses a two-sector model of endogenous growth with two common features of economic development: stages of sustained growth and underdevelopment traps. The model also demonstrates the transitional issues of a *temporary underdevelopment trap*, *seemingly sustainable growth*, and a *slowdown in productivity growth*. The temporary underdevelopment trap occurs when the economy exhibits a regime of extensive growth (i.e. slowly declining growth in physical capital with no growth in human capital) but then starts a transition to a sustained growth. The seemingly sustainable growth occurs when the economy exhibits a regime of intensive growth (i.e. both capitals are growing) but the growth of human capital ceases and the economy eventually finishes in a zero growth trap. The slowdown in productivity growth occurs when the transition from low growth stage to high growth stage is not monotonic.

Abstrakt

Prce se zabv dvousektorovm modelem se dvma charakteristickmi znaky ekonomickho rozvoje: stadiem trvalho rstu a past nerozvjejc se ekonomiky. Model tak demonstruje pechodov procesy *doasn pasti nerozvjejc se ekonomiky*, *zdnliv trvalho rstu* a *zpomalen rstu produktivity*. Doasn past nerozvjejc se ekonomiky se projev v ppadech, kdy se ekonomika rozvj zprvu extenzivn (t.j. fyzick kapitl roste klesajcm tempem a lidsk kapitl nulovm tempem), pak vak nhle nastane pechod k trvalmu rstu. Reim zdnliv trvalho rstu se projevuje u ekonomik, kter rostou intenzivn (t.j. roste fyzick i lidsk kapitl), pak se vak nhle zastav rst lidskho kapitlu a ekonomika konverguje do pasti s nulovm rstem. Zpomalen rstu produktivity se projevuje v ppadech, kdy pechod ze stdia s malm rstem do stdia s vysokm rstem nen monotonn.

Keywords: Two-Sector Growth Models, Economic Growth and Aggregate Productivity, Macroeconomic Analysis of Economic Development, Human Resources, Human Capital Formation

JEL Classification: O41, O40, O11, O15, J24

*I am deeply grateful to Elhanan Helpman,Kenneth Judd and Christian Keuschnigg for many helpful discussions. I also thank Randy Filer, Max Gillman, Stanley Reynolds, Viatcheslav Vinogradov as well as the participants in the Twelfth European Economic Association Conference in Toulouse, the Stanford Economics Department workshop and lunch seminar and CERGE-EI seminars for their helpful comments. Financial support from USAID and the Research Support Scheme of the HESP grant No.:184/1995 is appreciated.

I. Introduction

This paper analyses a two-sector model of endogenous growth with two features of economic development: stages of sustained growth and underdevelopment traps. By stages, we mean (in the sense of Rostow [1990]) regimes with persistent differences in sustainable growth rates. In particular, there is a *stage of low growth* before reaching an area of increasing returns. Then there is a take-off to a *stage of high growth*. By an underdevelopment trap, we mean when an economy exhibits a *stage of zero growth*. The model also demonstrates the transitional issues of a *temporary underdevelopment trap*, *seemingly sustainable growth*, and a *slowdown in productivity growth*. A temporary underdevelopment trap occurs when the economy exhibits a regime of extensive growth (i.e. slowly declining growth in physical capital with no growth in human capital) but then starts a transition to a sustained growth. Seemingly sustainable growth happens when the economy exhibits a regime of intensive growth (i.e. both capitals are growing) but the growth of human capital ceases and the economy eventually finishes in a zero growth trap. A slowdown in productivity growth occurs when the transition from low growth stage to high growth stage is not monotonic.

The model follows the Lucas-Uzawa learning-or-doing style (Lucas [1988]) with physical and human capital. We assume that there are positive externalities in the productivity of human capital[1], similar to Azariadis and Drazen [1990][2], since as Lucas states [in Lucas, 1988,p.19]: "... human capital accumulation is a social activity, involving groups of people in a way that has no counterpart in the accumulation of physical capital." We further assume that there is a frontier of 'theoretical knowledge' that is given exogenously and represents large advances in knowledge like an industrial revolution. The economy can approach the frontier via the education of people which facilitates adoption and implemention of new technologies. In this paper the technology diffusion leads to threshold or logistic types of externalities in the human capital accumulation process. Because of this, the human accumulation process can exhibit increasing returns to its inputs, depending on the average level of human capital.

The model is similar to the Zilibotti [1995] model except that ours uses two capital stocks as opposed to one and our engine of growth is human capital contrary to physical capital in "a Jones-Manuelli-type" production function. Our model is also different in that there are no indeterminacies, as in both the original Lucas model[1] and Zilibotti's model. And, in contrast to Zilibotti's model, where some kind of structural once-off shock is necessary to help to a better endowed economy to escape from a zero growth equilibrium, our model does not need any kind of "first movement" to start the growth engine.

Most of the papers on endogenous growth theory are restricted to steady state analysis. This is caused in part by the assumption that balanced growth regimes can serve as good approximations to the behavior of real economies. There are situations, however, such as wars, disasters, and the collapse of communist regimes in Central and Eastern Europe, when the relations between the levels of variables do not correspond to the steady state relations. Changes in government policies can also have this effect. In these economies there may appear, then, a transitional period during which they move back to steady states[4]. In such situations, short-run effects cannot be omitted and transitional dynamics have to be studied.

In this paper we go even further by claiming that in our suggested model or "theory of economic development"[5] the transition process can last a very long time, and in many cases the observations on an economy's behavior contain only transitional data. Therefore, if we want to model the behavior of such an economy we are forced to analyze and understand the transitional dynamics of the model. Moreover, the transitional dynamics of the model presented here are not reducible to the development of capital ratios as in the original Lucas model [see Mulligan and Sala-i-martin, 1992] and policy functions are general functions of both state variables.

The rest of the paper is organized as follows. In Section II. we develop the model and derive the first order conditions for decentralized economy equilibrium. In Section III. we use a step-like approximation of the "learning curve" with such a reformulation of the model that enables us later to analyze the transitional dynamics qualitatively, without using numerical methods. Section IV. is devoted to the analysis of steady states and to selected aspects of the model's comparative statics. The stages of low and high growth and of take-off are the content of Section V.. Section VI. continues the qualitative analysis of behavior and studies the existence of underdevelopment traps. We also use the results of a numerical simulation of the model calibrated to US data to demonstrate the mechanism of productivity slowdown as a result of the transition to the stage of higher balanced growth path.

II. The Model

Following the Lucas-Uzawa framework we assume a two-sector economy with a goods sector and an education sector. We further suppose that there are two ways technology innovation can occur: (i) large discontinuous advances which coincide with important eras like industrial revolutions and (ii) more cumulative and continuous progress during which the society learns how to use this potential. Similarly to Zilibotti [1995], we will consider the former as being exogenous in the sense that the economic activity has no effect on the occurrence of revolutionary advances. The second type of innovation depends on the gap between the present level of technology and the frontier productivity level given by the first type of innovation [see Nelson and Phelps, 1966]. We assume here, consistent with Lucas [1988] and Azariadis and Drazen [1990], that technical progress is driven by investment in human capital[6] such that:

$$\frac{\dot{B}_t}{B_t} = \psi \frac{B_H - B_t}{B_H} \dot{H}_t \tag{1}$$

where B_H means the frontier productivity, $B_H \geq B_t$, $\psi > 0$ is a parameter of the speed of difusion and H is the average level of human capital in the economy[7]. We can see that the farther an economy is from the frontier, the faster is the growth of productivity for a given level of investment. After solving equation (1) we obtain the following logistic solution

$$B(H) = \frac{B_H}{1 + (\frac{B_H}{B_0} - 1)e^{-\psi H}} \tag{2}$$

where B_0 is the initial level of productivity related to zero level of human capital. We can easily see from (2) that there is an upper bound of productivity given by B_H (i.e. if H goes to infinity productivity converges to B_H). Using (2) we will generalize the linear Uzawa-Rosen[8] formulation of the production function for human capital assuming that the level of productivity in the education sector, B, depends on the developmental level of a society expressed by the average level of human capital

$$\dot{h} = B(H)(1 - u)h \tag{3}$$

where $B(H)$ is given by (2). Therefore, the production function in the education sector (see equation (3)) exhibits increasing returns to all inputs at the social level for non-constant levels of productivity, i.e. $B'(H) \neq 0$, and constant returns to private inputs for any level of productivity.

Suppose now that the economy contains a constant number, N, of identical, infinitely-lived workers, which we will normalize to 1. All workers have the same skill level h and devote a fraction u of their (non-leisure) time to current production, and the remaining $1-u$ to human capital accumulation in the education sector. The effective labor input in production is then $l = uh$. In maximizing their life-time utility, the workers seek an optimal life-time consumption and working pattern $\{c_t, u_t\}$ which they achieve through the appropriate accumulation of financial and human wealth, a and h respectively:[9]

$$\max_{\{c_t,u_t\}} \int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-\theta}-1}{1-\theta}\right) dt \quad \text{s.t.} \tag{4}$$

$$\dot{a} = ra + wuh - c \tag{5}$$

$$\dot{h} = B(H)(1-u)h \tag{6}$$

$$\lim_{t\to\infty} a_t e^{-\int_0^t r_s ds} \geq 0 \tag{7}$$

$$0 \leq u \leq 1 \qquad a_0 > 0, \quad h_0 > 0, \tag{8}$$

The agents are endowed with perfect foresight with respect to future wages and interest rates, w and r. They know the production for creating new knowledge in the education sector. They cannot, however, influence the average level of accumulated knowledge, H. The price of the consumption good is normalized to one, ρ is the time preference parameter and σ is the intertemporal elasticity of substitution ($\theta = \sigma^{-1}$ is the degree of relative risk aversion). At each date, agents are endowed with a unit of time and a stock of financial and human wealth resulting from past accumulation. They choose consumption and allocate time for productive and educational purposes. Equation (7) refers to the no-Ponzi-game condition.

Using the Pontryagin Maximum Principle, we obtain the following first order conditions describing the agent's optimal choices, where λ and μ indicate the current-value shadow prices associated with financial and human wealth, respectively:

$$\lambda = c^{-\theta} \tag{9}$$

$$\lambda wh = \mu B(H)h \tag{10}$$

$$\dot{\lambda}/\lambda = \rho - r \tag{11}$$

$$\dot{\mu}/\mu = \rho - B(H) \tag{12}$$

$$\lim_{t\to\infty} k_t\lambda_t e^{-\rho t} = 0 \tag{13}$$

$$\lim_{t\to\infty} h_t\mu_t e^{-\rho t} = 0. \tag{14}$$

Equation (9) gives the condition where the maximizing agent is indifferent between consuming another unit of the good or saving it in the form of physical capital because the return from consumption (marginal utility) is the same as the return on investment in physical capital (shadow value λ). The next equation (10) states that the marginal return to study must be equal to the marginal return to work. The last two equations (11) and (12) describe the development of the shadow prices of capital. The growth rate of the shadow value of a particular type of capital is given by the gap between the discount rate and the net return on that capital. We can see, then, that the net return on financial capital is given by the interest rate r and the net return on human capital by $B(H)$.

On the production side, the economy consists of a large number of identical firms with a constant returns to scale Cobb-Douglas production function $F(k,l) = Ak^{\alpha}l^{1-\alpha}, \quad 0 < \alpha < 1$ where k is physical capital and l is labor expressed in effiency units. Each profit maximizing firm makes a static decision on how much labor and physical capital to rent from the agents:

$$\max_{k,l} \pi = F(k,l) - wl - (r+\delta)k \tag{15}$$

where δ is the rate of depreciation of physical capital. The maximization of (15) gives us the inverse factor demand functions:

$$r = F_k - \delta, \qquad w = F_l \tag{16}$$

where $F_k \equiv \partial F/\partial k$ and $F_l \equiv \partial F/\partial l$[10].

All markets clear in equilibrium. With physical capital being the only financial asset, $a = k$ and the average level of human capital is in equilibrium $H = h$. Thus the decentralized equilibrium is given by

$$\frac{\dot{k}}{k} = \frac{F}{k} - \delta - \frac{c}{k} \tag{17}$$

$$\frac{\dot{h}}{h} = B(h)(1-u) \tag{18}$$

$$\frac{\dot{c}}{c} = \sigma\left(F_k - \delta - \rho\right) \tag{19}$$

$$\frac{\dot{u}}{u} = (B(h)+\delta)\frac{1-\alpha}{\alpha} + B(h)u - \gamma(h)\frac{B(h)}{\alpha}(1-u) - \frac{c}{k} \tag{20}$$

$$k_0 > 0, \qquad h_0 > 0 \tag{21}$$

$$\lim_{t\to\infty} k_t\lambda_t e^{-\rho t} = 0, \qquad \lim_{t\to\infty} h_t\mu_t e^{-\rho t} = 0 \tag{22}$$

where $\gamma(h) = \mathrm{d}\ln B(h)/\mathrm{d}\ln h = B'(h)h/B(h)$ is the elasticity of total productivity in the education sector with respect to human capital.

The only difference between our model and the results of the Lucas-Uzawa model is the occurrence of the terms with $B(h)$ and variable elasticity $\gamma(h)$ in equation (20) for the behavior of time devoted to work u. This property prevents the standard procedure of finding a reduced form model with transformed variables exhibiting zero steady state growth. The nonexistence of such a reduced form makes the problem of obtaining a solution difficult not only from the analytical but also from the computational point of view. Therefore, we suggest an approximation of the model in the following section that will facilitate the analysis of its behavior.

III. Approximation of the Model

The aim of this paper is to describe different stages of development in the model caused by different initial conditions and different values of the parameters. To fulfill this task we suggest in this section *a 'qualitative' approximation*[11] of the model[12]. There are two good reasons for such an approximation. The first one relates to the difficulties of the analysis of transitional dynamics that were mentioned at the end of the last section. The second reason arises naturally from the fact that we want to describe transitional dynamics qualitatively as a development through different stages, where each stage can be characterized by distinct features.

It follows from the discussion of equation (20) in last section that it is the 'learning' curve (see Figure I) which complicates the analysis of the model. Thus any approximation of this curve which significantly simplifies the analysis and at the same time preserves the main features of the model behavior is beneficial. We claim a step-like approximation of the learning curve can do this job very well. Let's define $B(h)$ as

$$B(h) = \begin{cases} B_L, & 0 < h < \hat{H} \\ B_H, & h \geq \hat{H} \end{cases} \tag{23}$$

where $B_H > B_L$. Therefore, we assume that there exists a critical value of the average level of human capital, $\hat{H}$, such that below this level the productivity of the education sector is low (B_L), while above this level the productivity of the education sector is high (B_H) (see Figure II). The qualitative approximation[13] goes in two directions. The first one captures the fact that changes in the level of human capital have a negligible effect on productivity when the level of human capital is sufficiently low or sufficiently high. Secondly, the area in which increasing returns in the education sector are relevant is relatively narrow. This is consistent with the jump in (23).

The advantage of such a description of productivity is that it is piece-wise constant. Thus it enables us to split the development of the productivity parameter into three stages: low and high stages of development characterized by B_L and B_H, respectively, and the stage of 'take-off' collapsed into the switch from B_L to B_H when the average level of human capital in the economy reaches critical level $\hat{H}$. A discontinuous increase in the productivity of human capital at $\hat{H}$ will induce the agents to suddenly increase the time they spend on education. The negative jump in the time devoted to work (u) can be confirmed by a look at equation (20), which reveals that the discontinuity in productivity implies that the elasticity of productivity with respect to human capital $\gamma(\hat{H})$ is infinite at $\hat{H}$ and, thus, that the rate of growth of u is minus infinity at this point. The violation of the standard assumptions regarding the continuity of the production functions, however, creates problems with the use of the formulation of the problem given in (4)-(8) together with the implied necessary conditions expressed in (17)-(20).

These shortcomings can be overcome, however, by transforming the problem into a two-stage optimal control problem. During the first stage, period $[0, T)$, the representative agent maximizes his lifetime welfare $V(k_0, h_0)$ subject to the relationships found in the economy having low productivity in the education sector B_L. Lifetime welfare is expressed in (24) as the welfare function over period $[0, T)$ together with the scrap value $V_{II}(k_T, h_T)e^{-\rho T}$ given by the welfare function of the second stage discounted to time 0. The first stage finishes at time T when the average level of human capital reaches the critical amount $\hat{H}$ and productivity jumps to B_H. Thus during the second stage the agent maximizes his welfare given by $V_{II}(k_T, h_T)$ in (29) on the interval (T, ∞) subject to conditions in an economy with high productivity in education sector.

$$V(k_0, h_0) = \max_{\{c_t, u_t\}} \left\{ \int_0^T e^{-\rho t} \left(\frac{c_t^{1-\theta} - 1}{1-\theta} \right) dt + V_{II}(k_T, h_T) e^{-\rho T} \right\} \tag{24}$$

subject to

$$\dot{k} = rk + wuh - c \tag{25}$$

$$\dot{h} = B_L(1-u)h \tag{26}$$

$$0 \leq u \leq 1 \qquad k_0 > 0, \quad h_0 > 0 \tag{27}$$

$$(28) \qquad h_T = \hat{H} \qquad k_T \text{ is free} \qquad T \text{ is fixed,}$$

with

$$(29) \qquad V_{II}(k_T, h_T) = \max_{\{c_t, u_t\}} \int_T^\infty e^{-\rho(t-T)} \left(\frac{c_t^{1-\theta} - 1}{1-\theta} \right) dt$$

subject to

$$(30) \qquad \dot{k} = rk + wuh - t - c$$

$$(31) \qquad \dot{h} = B_H(1-u)h$$

$$(32) \qquad 0 \leq u \leq 1 \qquad k_T > 0, \quad h_T = \hat{H}$$

$$(33) \qquad \lim_{T\to\infty} k_t e^{-\int_T^t r_s ds} \geq 0.$$

Using the Pontryagin Maximum Principle of optimal control we get the following necessary conditions for our two-stage optimal control problem

$$(34) \qquad \lambda_I = c^{-\theta} e^{-\rho t}$$

$$(35) \qquad \lambda_I wh = \mu_I B_L h$$

$$(36) \qquad \dot{\lambda}_I / \lambda_I = -r$$

$$(37) \qquad \dot{\mu}_I / \mu_I = -B_L$$

$$(38) \qquad \lambda_{I,T} = \frac{\partial V_{II}(k_T, h_T)}{\partial k_T} e^{-\rho T}$$

for $t \in [0, T)$ and

$$(39) \qquad \lambda_{II} = c^{-\theta} e^{-\rho(t-T)}$$

$$(40) \qquad \lambda_{II} wh = \mu_{II} B_H h$$

$$(41) \qquad \dot{\lambda}_{II} / \lambda_{II} = -r$$

$$(42) \qquad \dot{\mu}_{II} / \mu_{II} = -B_H$$

$$(43) \qquad \lim_{t\to\infty} k_t \lambda_{II,t} = 0$$

$$(44) \qquad \lim_{t\to\infty} h_t \mu_{II,t} = 0$$

for $t \in (T, \infty)$. Equation (38) is the transversatility condition for free endpoint with scrap value [see e.g. Kamien and Schwartz, 1991] and λ_I, μ_I and λ_{II}, μ_{II} are the present-value costate variables related to the first and second stage problem respectively.

The initial and TVC conditions given by (27) and (43)-(44), respectively, are straightforward. The TVC condition for connecting the two stages is (38) and there is no condition on the development of μ_{II}. The competitive equilibrium in the model variables is then given by the following proposition.

Proposition 1 *(Jump) The representative agent problem given by (24)-(33) with the productivity parameter $B(H)$ characterized along (23) leads to the first order necessary conditions (34)-(38) for $t \in [0,T)$ and (39)-(44) for $t \in (T,\infty)$. Thus the competitive equilibrium dynamics of the two-stage problem (24)-(33) can be expressed by the following two systems of equations*

$$\frac{\dot{k}}{k} = \frac{F}{k} - \delta - \frac{c}{k} \tag{45}$$

$$\frac{\dot{h}}{h} = B_I(1-u) \tag{46}$$

$$\frac{\dot{c}}{c} = \sigma\left(F_k - \delta - \rho\right) \tag{47}$$

$$\frac{\dot{u}}{u} = (B_I+\delta)\frac{1-\alpha}{\alpha} + B_I u - \frac{c}{k} \tag{48}$$

where $I \in \{L,H\}$ with further initial, 'connecting' and TVC conditions

$$k_0 > 0, \qquad h_0 > 0 \tag{49}$$

$$k_{T_-} = k_{T_+}, \qquad h_{T_-} = h_{T_+} = \hat{H}, \qquad c_{T_-} = c_{T_+}, \qquad u_{T_-} > u_{T_+} \tag{50}$$

$$\lim_{t\to\infty} k_t\lambda_t = 0, \qquad \lim_{t\to\infty} h_t\mu_t = 0, \tag{51}$$

respectively.

Proof: (see Appendix).

The piece-wise constancy of the productivity parameter enables us to transform the model into two reduced models, related to two productivity levels (B_H and B_L), expressed in transformed variables[14]: the physical to human capital ratio $x \equiv k/h$, the consumption to physical capital ratio $q \equiv c/k$ and time devoted to work u. These new variables have the convenient property of zero growth rates in the steady state that facilitates further analysis.

By applying the suggested transformations to equations (45)-(48), the model can be expressed as the following two systems of equations:

$$\frac{\dot{x}}{x} = A\left(\frac{u}{x}\right)^{1-\alpha} - q - \delta - B_I(1-u) \tag{52}$$

$$\frac{\dot{q}}{q} = (\sigma\alpha - 1)A(\frac{u}{x})^{1-\alpha} - \sigma\rho + (1-\sigma)\delta + q \tag{53}$$

$$\frac{\dot{u}}{u} = \frac{(B_I + \delta)}{\alpha} - \delta - (1-u)B_I - q \tag{54}$$

with $I \in \{L, H\}$ together with the initial, connecting, and TVC conditions given by (49)-(51).

IV. Balanced Growth Path

If we look at equation (46) for the growth of human capital we can see that the property of locally increasing returns, caused by an upper bound on the externality effect, is critical for the existence of a balanced growth path (BGP). With globally increasing returns, the model would exhibit an ever-accelerating growth[15].

Each of the two systems implies, analogously to the original Lucas-Uzawa model[16], a unique BGP with zero growth in the transformed variables. Both capitals and consumption will grow along the BGP at positive growth g_L^* and g_H^* for productivity B_L and B_H, respectively. By applying the condition of BGP to equations (17)-(20) we obtain the following:

$$g_I^* = \sigma(B_I - \rho) \tag{55}$$

$$u_I^* = 1 - \frac{\sigma(B_I - \rho)}{B_I} \tag{56}$$

$$q_I^* = \frac{\delta + B_I}{\alpha} - \sigma(B_I - \rho) - \delta \tag{57}$$

$$x_I^* = \left(\frac{\delta + B_I}{\alpha A}\right)^{\frac{1}{\alpha-1}} u_I^* \tag{58}$$

where $I \in \{L, H\}$.

Equation (55) says that consumption, physical and human capital grow with a positive balanced growth rate $g_I^* > 0$ only if productivity in the education sector is sufficiently high and/or people are not too impatient. Combined with equation (56), we can see that a positive growth rate is possible only if some fraction of the endowed time is spent on education $u_I^* < 1$[17]. Interestingly, for an economy with a high degree of relative risk aversion θ (low intertemporal elasticity of substitution) where people prefer to smooth the consumption path, the resulting balanced growth rate will be lower while a thriftier and more patient society will enjoy higher growth rates. Equation (58) is based on the fact that at the steady state net returns from both capitals are identical, i.e. $F_k^* - \delta = B_I$. Moreover, we can see that the parameters which increase returns to physical capital, i.e. productivity A and capital share α, also increase the capitals ratio.

Because of the presence of externalities in the education sector, the sensitivity of the model to changes in the productivity of human capital B_I will be very important. Therefore, we take the derivatives of the steady state values with respect to B_I

$$\frac{\partial g_I^*}{\partial B_I} = \sigma > 0 \tag{59}$$

$$\frac{\partial u_I^*}{\partial B_I} = -\frac{\rho\sigma}{B_I^2} < 0 \tag{60}$$

$$\frac{\partial q_I^*}{\partial B_I} = \frac{1}{\alpha} - \sigma = \begin{cases} > 0, & \alpha < \sigma^{-1} \\ = 0, & \alpha = \sigma^{-1} \\ < 0, & \alpha > \sigma^{-1} \end{cases} \tag{61}$$

$$\frac{\partial x_I^*}{\partial B_I} = -\left(\frac{\alpha A}{\delta + B_I}\right)^{\frac{1}{1-\alpha}} \left[\frac{u_I^*}{(1-\alpha)(B_I+\delta)} + \frac{\sigma\rho}{B_I^2}\right] < 0 \tag{62}$$

Equation (59) states that the higher the productivity of the education sector is, the higher the balanced growth rate of an economy will be. The growth increases as society's willingness to substitute today's consumption for tomorrow's increases (i.e. σ is higher). Equation (60) demonstrates that in an economy with a more effective education sector people study more and work less. This effect is stronger the less patient people are (bigger ρ) and the more they are willing to substitute consumption across time, while the effect diminishes with the productivity of the education sector. Therefore, not surprisingly, human capital will be more abundant relative to physical capital on the BGP if the education sector is more productive as we can see from (62). The dependence of the consumption-to-physical-capital ratio on the effectiveness of the education sector given by (61) is non-monotonic. In an economy where people strongly prefer to smooth consumption (i.e. the intertemporal elasticity of substitution is smaller than the inverse of the capital share parameter $< \alpha^{-1}$) the ratio of consumption to physical capital will be higher when the productivity of the education sector is higher. Notice that a higher B_I means relatively less abundant physical capital and therefore $(c/k)^*$ has to rise to keep consumption at the same level. On the contrary, an economy with a low willingness to smooth consumption ($\sigma > \alpha^{-1}$) will end up with a lower ratio of consumption to physical capital.

When $I = H$, the above equations (55)-(58) characterize the BGP related to high productivity B_H, when $I = L$, the BGP is related to low productivity B_L. However, the global behavior of the model is such that the low BGP will never be reached even if we allow for infinite time because a growing economy will always reach the region of higher productivity in finite time. The whole system, therefore, has only one global BGP, the path related to the high productivity level. The BGP related to lower productivity is a fictive BGP or *quasi-balanced growth path* (QBGP) which works only as a temporary attractor in (k, h) space.

V. Transitional Dynamics I: Stages of Growth

Using the above results we can divide the transitional dynamics into three stages: *the stage of low growth* before the productivity miracle occurs when productivity is low, $B(h) = B_L$, and the economy develops according to (52)-(54); *the stage of take-off* when the miracle happens and the economy switches from low productivity of education sector to high productivity; and *the stage of high growth* occurs in the aftermath of the productivity miracle when productivity is high, $B(h) = B_H$, and the economy develops according to (52)-(54).

In order to study the transitional dynamics of the model it is useful to know more about the local behavior around the BGPs. The standard approach is to derive a log-linear approximation of the model around the BGP. Taking the first-order Taylor expansions of equations (52)-(54) in logarithmic variables $\tilde{x} \equiv \ln x, \tilde{q} \equiv \ln q$, and $\tilde{u} \equiv \ln u$ at the particular steady states, we obtain the following system of linear differential equations expressed in matrix form (see Appendix A)

$$\begin{bmatrix} \dot{\hat{x}}_I \\ \dot{\hat{q}}_I \\ \dot{\hat{u}}_I \end{bmatrix} = \begin{bmatrix} -\varepsilon_I & -q_I^* & q_I^* \\ -(1-\sigma\alpha)\varepsilon_I & q_I^* & (1-\sigma\alpha)\varepsilon_I \\ 0 & -q_I^* & B_I u_I^* \end{bmatrix} \begin{bmatrix} \hat{x}_I \\ \hat{q}_I \\ \hat{u}_I \end{bmatrix} \tag{63}$$

where $\varepsilon_I = -\frac{(1-\alpha)}{\alpha}(B_I + \delta) < 0$, $\hat{x}_I \equiv \tilde{x} - \tilde{x}_I^*$, $\dot{\hat{x}}_I \equiv \dot{\tilde{x}}$ and $I \in \{L, H\}$. This system together with the accompanying initial, connecting and final conditions (49)-(51) serves as a piece-wise log-linear approximation of the model.

As is shown in Appendix A, each system has one negative and two positive eigenvalues, $\lambda_{I,1} = \varepsilon_I < 0$, $\lambda_{I,2} = q_I^* > 0$, and $\lambda_{I,3} = B_I u_I^* > 0$ again with $I \in \{L, H\}$. Thus the system with two control, q, u, and one state variable, x, is saddle-path stable[18].

Stage of High Growth If the logistic function has the step-like shape given by (23), then any economy whose average level of human capital is higher than the critical level $\hat{H}$ will have the productivity of the education sector given by B_H. The TVC conditions given in (51) imply that the economy will move along the saddle path related to the high BGP. According to the approximation of the behavior along the saddle path given in Proposition 6 in Appendix A, we can conclude that the policy functions for u and q are both either upward-sloping, if the share of physical capital in the goods production function α is larger than the inverted value of the intertemporal elasticity of substitution σ^{-1}, or downward-sloping, if the share of physical capital in the goods production function α is smaller than the inverted value of the intertemporal elasticity of substitution σ^{-1}.[19] Similar to the discussion in Section IV., we consider the latter case when the product of capital share and the intertemporal elasticity of substitution is smaller than one as more realistic and therefore we examine only this case in the rest of the paper.

Low Growth Stage The stage of low growth occurs before the productivity miracle happens at time T. For the analysis of transitional dynamics during the stage of low growth, the connecting conditions given in (50), instead of the TVC conditions, are relevant. As we already discussed, the connecting conditions are such that at the moment of the miracle there will be no jump in consumption (i.e. the projection of the path of the economy expressed in (x, q, u) onto (q, x) plane must lie, at the moment T, on the projection of the saddle path related to high productivity). Because there is no connecting condition regarding the time devoted to work, there will be a negative jump in working time at the moment of the miracle. This means that the only adjustment that takes place before the miracle is the adjustment in consumption. Thus the relation between the consumption-capital-ratio and the capitals ratio is not given by the projection of the saddle path onto (q, x) plane contrary to the relationship between the time devoted to work and the capitals ratio which is given by the projection of the saddle path onto (u, x) plane.

Take-off Stage The take-off stage collapses to the jump in total factor productivity in the education sector from the low level B_L to the high level B_H. According to the above discussion, the time devoted to work jumps down from the saddle path related to B_L to the saddle path related to high productivity B_H at the moment of the productivity miracle.

Using the characterizations of the BGP and QBGP in (55)-(58) we can define *a generalized balanced growth path* (GBGP) as a path which connects the QBGP related to the stage of low growth with the BGP related to the stage of high growth. From the analysis of steady states we know that the slope of the GBGP is determined by the QBGP and the BGP ratios of physical to human capital ratio in the particular regions. The GBGP in (k, h) is depicted in Figure III where the adjustment path of capitals between the QBGP and the BGP can be seen. If the economy is initially off the BGP or the QBGP it will first move to the GBGP and then along it. Thus the transitional dynamics of the model can be decomposed into two kinds of transitions: (i) the transitions arising due to imbalances in the levels of the stock of capital between the two sectors with respect to the particular steady state (similarly to the original Lucas model)[20] and (ii) the transitions related to the level of human capital with respect to its critical level.

A. Development through Stages

Using the understanding regarding the transitions derived above, we can summarize the transitional behavior of an economy for different initial levels of human and physical capital. We will show them in the projection of the phase plane (x, q, u) into the (x, u) plane in Figure IV. There is a 'fictive' steady state L related to the low growth g_L^* with a stable downward-sloping saddle path $U(x; B_L)$ and a steady state H related to the high growth rate g_H^* with saddle path $U(x; B_H)$. Steady state L is located to the northeast of steady state H as it follows from the results of the effects of a change in productivity given by (55)-(58) in Section IV. where higher productivity in the education sector leads to a relatively higher level of human capital $\tilde{x}_H^* < \tilde{x}_L^*$ and more time spent in school $\tilde{u}_H^* < \tilde{u}_L^*$. According to (80) and (86) in Appendix A, the saddle paths are downward sloping when the intertemporal elasticity of substitution is smaller than the inverse of the capital share[21], $\sigma < \alpha^{-1}$, with the slope decreasing as the productivity increases. As was described in Mulligan and Sala-i-martin [1992] the downward slope of the policy functions is caused by the fact that people with a strong desire to smooth consumption prefer, when they are poor (and have a low level of physical capital), to build their physical capital through increased work effort rather than through increased savings. Thus as the level of physical capital increases ($\tilde{x}$ grows) they work less ($\tilde{u}$ declines). Let us examine now two typical transitions.

A less developed country with an initial relative lack of human capital Assume that the economy is initially more abundant in physical capital $x_H^* < x_L^* < x_1(0)$ but less developed $h_1(0) < \hat{H}$. Because $k_1(0)/h_1(0) > (k/h)^*$ the return to human capital exceeds that to physical capital, motivating people to spend more time on education and thus human capital grows faster than physical capital. Because the level of human capital is lower than the critical value $\hat{H}$, the productivity of the education sector is low (B_L) and thus the economy is initially at point A and moves along saddle path $U(\tilde{x}, B_L)$ toward the 'fictive' steady state L, related to the low productivity of education. If the level of human capital was initially much lower than $\hat{H}$ the economy would approach very close to L. This is what we call *the stage of low growth* rate g_L^*. As the economy continues to develop, the level of human capital grows. Thus, at some finite time the economy will approach the critical level of human capital at point B. The sudden increase in the productivity of the education sector in turn increases the return to human capital, motivating people to increase the time devoted to study. Therefore u jumps to point C on the high productivity saddle path, *the 'take-off' stage*[22]. Afterwards, the economy, with a more productive education sector, continues to move along the saddle path $U(\tilde{x}, B_H)$ to the new steady state H with the high growth rate g_H^*, that characterizes *the stage of high growth.*

A less developed country with initially relatively less abundant physical capital Consider now an economy that has initially relatively less abundant physical capital $x_2(0) < x_H^* < x_L^*$ and is again less developed $h_1(0) < \hat{H}$. Symmetrical to the above case (because $k_1(0)/h_1(0) < (k/h)^*$) the return on physical capital is higher than that of human capital and people work harder in order to accumulate physical capital faster than human capital. Because the level of human capital is lower than critical value $\hat{H}$, the productivity of the education sector is low given by B_L and thus the economy is initially at point D and moves from it along saddle path $U(\tilde{x}, B_L)$ toward the 'fictive' steady state L. If the level of human capital is initially much lower than $\hat{H}$, the economy will move relatively very close to L, *the stage of low growth.* As the economy continues to develop, the level of human capital grows. It is again only a question of time before the economy manages to reach the critical level of human capital at point E. The sudden increase in the productivity of the education sector increases the return to human capital, motivating people to increase the time devoted to study. Therefore u jumps to point F on the high productivity saddle path, *the 'take-off' stage*. Afterwards, the economy with a more productive education sector continues to move along the saddle path $U(\tilde{x}, B_H)$ to the new steady state H with the high growth rate g_H^* characterizing *the stage of high growth*.

If the level of human capital is initially low, the stage of low growth is prolonged (the transition via E) and the economy with low incentives to study gradually loses its advantage in relatively more abundant human capital. After reaching the critical level of human capital the increased return on education motivates people to study more. This increased investment in education causes a decline in $\tilde{x}$ from F to H. Thus there is an overshooting in the relative level of physical to human capital during the whole transition from D to H. On the other hand, if the economy is initially not very far from the critical level of human capital the productivity jump will happen before the economy converges to L. In this case there will be no (or very little) overshooting during the transition from D via E' and F' to the new steady state H.

VI. Transitional Dynamics II: Underdevelopment Traps and Productivity Slowdown

A. Extensive Growth and Underdevelopment Traps

In setting up our model we assumed that the time endowment for agents in the economy was equal to 1. We will show below that such situations can occur where the agents would be willing to spend more time working than they have available. This will cause the imposed constraint $u \leq 1$ to bind. As a result no time is devoted to education and, therefore, the engine of endogenous growth is stopped with adjustments only in physical capital. We call this development *the stage of extensive growth*. Whether this situation is permanent and the economy gradually moves to a stage of zero growth (i.e. an underdevelopment trap) or it is only temporary and growth of human capital reappears after a certain period of time, will be analyzed in this section.

After imposing the constraint $u \leq 1$ on the model, we obtain the following conditions instead of equation (10):

$$\lambda w h \geq \mu B(h) h \text{ and } (1-u) \geq 0. \quad (64)$$

The necessary condition for the stage of extensive growth, when there is no accumulation in human capital, is stated in the following Proposition.

Proposition 2 *(Extensive Growth) If an economy is in the situation where the return to study is lower than the return to work ($(1-\alpha)Ax^{\alpha}\lambda > \mu B_L$), then nobody is willing to study, $u = 1$, and the only capital which adjusts is physical capital.*

The Proof is straightforward from (64).

When the halt to human capital accumulation during extensive growth is permanent the transition process finishes at zero growth steady state and the economy is caught in the underdevelopment trap. The necessary condition for the existence of such an underdevelopment trap is that the education efficiency parameter is smaller than the discount rate $B_L < \rho$ as Proposition 3 below claims.

Proposition 3 *(Underdevelopment Trap) An economy can be trapped in the stage of zero growth only if $B_L \leq \rho$.*

Proof: See the Appendix.

Whether an economy really falls into an underdevelopment trap depends on the initial conditions for physical and human capital, in addition to the above necessary condition. As shown in Propositions 4 and 5 below, we can distinguish two transitions which result in an underdevelopment trap. The first is the transition to an underdevelopment trap via the process of extensive growth depicted in Figure V when the economy is initially not very developed, i.e. $h_1(0) < \hat{H}$, and has relatively more abundant human capital than physical capital ($\tilde{x}_1(0) < \tilde{x}_L^*$). These two conditions imply that the returns to education are much lower than those to work and people have an incentive to work very hard, $u > 1$, (see point A'). Because of the constraint on their time endowment, however, they cannot spend more time at work than $u = 1$. Spending all their time at work enables them to produce enough to cover their consumption and invest their savings in physical capital. The growing physical capital leads to diminished returns because the effectiveness of labor is fixed. Despite the declining returns people are still motivated to save (till $r > \rho$) and only to work because the returns to the investment in education are even lower ($B_L < \rho$). Thus the economy starts at A and moves to the underdevelopment steady state U with a zero growth rate.

A similar behavior occurs when the economy is initially not very developed ($h_1(0) < \hat{H}$) and has relatively more abundant physical capital ($\tilde{x}_1(0) < \tilde{x}_L^*$) but the level of physical capital is still low enough $\tilde{x}(0) \leq \tilde{x}_C$ that the return to study is smaller than the return to work and nobody wants to study. The return on physical capital in this economy with a fixed level of human capital is so small that $r - \rho < 0$ and people have an incentive not to save but instead to consume their physical capital. This situation will not end until the economy reaches its steady state U with zero growth. The necessary and sufficient conditions for the transition to an underdevelopment trap via extensive growth are provided by the following proposition.

Proposition 4 *(Underdevelopment Trap with Extensive Growth) The economy will reach an underdevelopment trap via adjustments only in physical capital if, and only if, the steady state growth related to low education efficiency is not positive and the initial capital ratio* $\tilde{x}(0) \leq \tilde{x}_C$ *for any level of initial human capital lower than the threshold value* $h(0) < \hat{H}$*, where* $\tilde{x}_C$ *is such that* $U(\tilde{x}_C; B_L) = 0$*.*

Proof: See Appendix.

The second kind of transition mentioned above is the transition to an underdevelopment trap via intensive growth (see Figure V). This may happen only when an economy is initially low developed, i.e. $h_2(0) < \hat{H}$ and physical capital is relatively more abundant, with this abundance being sufficiently high $\tilde{x}_2(0) > \tilde{x}_C$. These two conditions imply that the return to human capital is initially higher than that to physical capital and people study and work in such a way that human capital grows faster than physical capital (point B). During the transition the difference between the two returns diminishes ($\tilde{u}$ is increasing) as the economy gets closer to its steady state. It will be trapped in a pattern of extensive growth followed by a zero growth steady state at U unless it manages to attain the critical amount of human capital before the accumulation of human capital ceases, i.e. before it reaches the stage of extensive growth with no education (point C),

$h_2(0) < H_C(x_2(0))$. We call the transition during which the economy initially exhibits sustainable, positive growth in both capitals, but is ultimately not successful in escaping from the underdevelopment trap *seemingly sustainable growth*. The necessary and sufficient conditions for the existence of the *underdevelopment trap with seemingly sustainable growth* are given in the following Proposition.

Proposition 5 *(Underdevelopment Trap with Seemingly Sustainable Growth) The economy will reach an underdevelopment trap preceded by positive growth in both types of capital if, and only if,* $g_L^* \leq 0$*, the initial capital ratio* $\tilde{x}(0) > \tilde{x}_C$ *and the initial level of human capital satisfies* $h(0) < H_C < \hat{H}$ *such that* $H_C(\tilde{x}(0)) = \inf\{h(0) : h(T) = \hat{H}$ *and* $\tilde{x}(T) = \tilde{x}_C\}$*.*

Proof: See Appendix.

If the quasi-steady state related to the lower productivity of the education sector B_L exhibits positive growth $g_L^* > 0$ an economy with a sufficient level of human capital relative to physical capital ($\tilde{x}_1(0) < \tilde{x}_C$) will experience a return to human capital that is much lower than the return to physical capital and people will have an incentive to work very hard, $u > 1$, (point A' in Figure VI). Again, because of the constraint on u, they cannot spend more time at work than $u = 1$ and thus the return to study is lower than the return to work resulting in people working full time. Thus the economy starts at A and exhibits a stage of extensive growth during which human capital is not accumulated. The increase in physical capital leads to a relative decline in the return to physical capital compared to that of human capital. This development also causes a decline in the return to work. Thus it is only a question of time until the return is so low that the economy starts to accumulate human capital again. The growth in the level of human capital makes it again only a question of time until the economy reaches the critical level of human capital and takes off. We, therefore, call this behavior of an economy initially exhibiting extensive growth but then eventually taking off to high growth rate a *'seemingly' or temporary underdevelopment trap.*

A summary of the transitional dynamics with underdevelopment traps is given in Figure VIII where the various transitions are depicted in plane (k, h). The main notations are the same as in Figure III. In addition line X_C splits the vertical area related to the less developed economy between 0 and $\hat{H}$ into two parts: above line X_C returns to study are equal to returns to work; below the line returns to study are lower than returns to work and the time constraint is binding. The shadowed area X_{UT} between line X_C and the vertical line going through $\hat{H}$ is the area of underdevelopment traps with extensive growth. As we can see the transition paths inside of these boundaries are vertical lines (i.e. there is no accumulation of human capital) converging to line X_L^* representing underdevelopment traps. The shadowed area X_{SS} above line X_C and below the bold line represents the area of seemingly sustainable growth. An economy with the initial conditions inside of X_{SS} will initially exhibit growth in both capitals till it reaches area X_{UT} of underdevelopment traps (see the transitions inside of X_{SS}). Analogously the behavior of the economy in a seemingly underdevelopment trap is demonstrated in Figure IX where shadowed area X_{SU} shows the area of extensive growth.

As demonstrated above, the relationship between the productivity parameter in the education sector and the subjective discount rate is critical for the existence of multiple equilibria or a quasi-steady state. It can be represented in the form of a "bifurcation diagram" shown in Figure VII which expresses the dependence of the growth rate at steady states on the subjective discount rate ρ[23] for the two given values of productivity in the education sector B_H and B_L.

Propositions 4 and 5 imply that there is one BGP equilibrium with a stage of high growth rate, and one quasi-balanced growth path equilibrium QBGP with a stage of low growth rate for low values of the time preference parameter, $\frac{(1-\sigma)\sigma}{(1-\sigma)\sigma+1}B_H < \rho < B_L$, and two BGP equlibria (high growth rate and zero growth rate) for high values of ρ, $B_L \leq \rho < B_H$. BGP equilibria are pictured as solid lines and QBGP as bold dashed line in Figure VII. There is, therefore, one bifurcation point, $\rho = B_L$, at which the economy is structurally unstable and the qualitative behavior of the model changes.

Imagine now that there appears a new productivity miracle (say a scientific revolution) such that $B_{HH} > B_H$. What would this imply for the existence of multiple equlibria, underdevelopment traps and quasi-steady states? Clearly, the solid downward sloping line of the balanced growth path in Figure VII with growth g_H^* would change to a dashed line implying that the stage of high growth rate is a quasi-steady state instead of a steady state. Thus, for $\frac{(1-\sigma)\sigma}{(1-\sigma)\sigma+1}B_{HH} < \rho < B_L$, there would be two quasi-steady states related to growth rates g_L^* and g_H^* and one steady state with the very high growth rate g_{HH}^*. Two BGP equlibria, one with a very high growth rate and one with a zero growth rate and one QBGP with an intermediate, high growth rate, will appear for values of ρ, $B_L \leq \rho < B_H$. Ultimately, however, there will be only two BGP equlibria, one with very high growth and one with zero growth for values of ρ, $B_H \leq \rho < B_{HH}$.

B. Transition to the High Growth Stage and Productivity Slowdown

The last dynamic phenomenon is related to the transition from a lower growth stage to a higher growth stage which may be accompanied by *a temporary productivity slowdown.* This provides an optimistic explanation of the productivity slowdown observed in the United States in 70s and 80s in contrast to the traditional explanation derived from the neoclassical growth model[24] [see e.g. Bailey and Schultze, 1990]. To demonstrate this phenomenon which we are not able to detect from the qualitative analysis of the model, we need to numerically simulate the calibrated model.

Let us assume (as is the case at the end of previous section) that the United States is experiencing a new industrial revolution with the appearance of a new frontier for the productivity level B_{HH}[25]. We then calibrate the model using the typical values of parameters found in Lucas [1988] and Mulligan and Sala-i-martin [1993] (see Appendix C) with implied values of 2.8% for lower steady state growth and 4% for higher steady state growth.

Looking at the results of the simulation related to a lower value of the intertemporal elasticity of substitution shown in Figure X, we can observe several features of the transition. During the early phase of the take-off, at the beginning of the area of increasing returns in the education sector, the path of the economy goes through a narrow region where a temporary decline in the growth rate of physical capital occurs (i.e. growth undershooting in physical capital). The growth rate of human capital is already accelerating at that moment because the agents in the economy, at the moment of take-off, foresee much greater future returns to education and, therefore, study more and work less. However, they still prefer a smooth pattern of consumption because the intertemporal elasticity of substitution is low. This leads to a lack of savings and thus low investment in physical capital, causing a decline in the growth rate of physical capital. The increased accumulation of human capital means higher efficiency for the labor force. Thus people are able to produce more even with a lower fraction of time allocated to work, resulting in a general tendency for higher growth in the economy and only a temporary lack of investment in physical capital and stagnation of consumption growth. After the increasing returns end, there will be a slowdown in the growth rate of human capital (growth overshooting in human capital).

If the value of the intertemporal elasticity of substitution is high, the agents will be more willing to postpone consumption and there may even be a temporary decrease in the growth rate of consumption (see Figure XI). With the higher elasticity, the perceived future growth will also be higher and people will have a stronger incentive to study leading to an apparent decline in output growth, *a productivity slowdown*, together with an even more profound decline in the growth of physical capital[26]. Again the increased efficiency of the labor force soon outweighs all the negative tendencies in the economy and accelerating growth starts to be a general feature of the economy. After reaching the productivity frontier, the increasing returns are depleted and the economy converges to the new high steady state growth. The growth overshooting in human capital is again more profound when the intertemporal elasticity of substitution is higher, as can be seen in Figure XI.

VII. Conclusions

We have presented a two-sector endogenous growth model with threshold externalities in the process of human capital accumulation. This model can exhibit the two main phenomena of economic development: underdevelopment traps and sustained growth. Without analyzing the transitional dynamics of the model, not much can be determined with regard to its behavior. Even the study of corner solutions has a dynamic dimension in our case when we use an infinite life-time framework. This is the case for other dynamic phenomena such as a *temporary underdevelopment trap* and *seemingly sustainable growth*. *Temporary underdevelopment trap* is a situation when the economy exhibits a slowly declining growth in physical capital with no growth in human capital followed by a sudden transition to a sustained or quasi-sustained growth path. *Seemingly sustainable growth* occurs when the economy temporarily goes through a transition with positive growth of human capital but is finally trapped in zero growth stage. Another dynamic phenomenon, a *slowdown in productivity growth*, occurs when the transition from the low growth stage to the high growth stage is not necessarily monotonic and can exhibit a temporary decline in growth rates. Because of increasing returns in the education sector caused by the increasing effect of externalities, people spend relatively more time studying, and in improving their skills generally. Therefore, the growth of total productivity will decline. After some time, the returns from acquiring higher skills will reverse this decline and productivity will grow at a higher rate. Thus our model provides an optimistic explanation of the productivity slowdown observed in the 80s in the United States as a temporary phenomena. The optimism of this explanation lies in the prediction that the slowdown will be followed by a stage of higher growth.

Appendix A: Log-linearization of the Reduced Model

The system of differential equations for the reduced model is derived from Eqs. (52)-(54) by

$$\frac{\dot{x}}{x} = A(\frac{u}{x})^{1-\alpha} - q - \delta - B(1-u) \tag{65}$$

$$\frac{\dot{q}}{q} = (\sigma\alpha - 1)A(\frac{u}{x})^{1-\alpha} - \sigma\rho + (1-\sigma)\delta + q \tag{66}$$

$$\frac{\dot{u}}{u} = \frac{(B+\delta)}{\alpha} - \delta - (1-u)B - q. \tag{67}$$

We will log-linearize the model at the steady state (x^*, q^*, u^*) given from Eqs. (55)-(58) by

$$g^* = \sigma(B-\rho) \tag{68}$$

$$u^* = 1 - \frac{\sigma(B-\rho)}{B} \tag{69}$$

$$q^* = \frac{\delta+B}{\alpha} - \sigma(B-\rho) - \delta \tag{70}$$

$$x^* = \left(\frac{\alpha A}{\delta+B}\right)^{\frac{1}{1-\alpha}} u^*. \tag{71}$$

Taking a first-order Taylor expansion of Eqs. (65)-(67) in logarithmic variables $\ln x, \ln q$, and $\ln u$ we obtain

$$\frac{d\ln x}{dt} \approx -A(1-\alpha)(\frac{u^*}{x^*})^{1-\alpha} d\ln x + q^* d\ln q + (A(1-\alpha)(\frac{u^*}{x^*})^{1-\alpha} + Bu^*) d\ln u \tag{72}$$

$$\frac{d\ln q}{dt} \approx -(1-\alpha)(\sigma\alpha-1)A(\frac{u^*}{x^*})^{1-\alpha} d\ln x + q^* d\ln q +$$
$$+(1-\alpha)(\sigma\alpha-1)A(\frac{u^*}{x^*})^{1-\alpha} d\ln u \tag{73}$$

$$\frac{d\ln u}{dt} \approx -q^* d\ln q + Bu^* d\ln u. \tag{74}$$

If we introduce the notation $\tilde{x} \equiv \ln x$, $\dot{\tilde{x}} \equiv \frac{d}{dt}(\ln x)$, $\hat{x} \equiv \tilde{x} - \tilde{x}^*$ and $\dot{\hat{x}} \equiv \dot{\tilde{x}} - \dot{\tilde{x}}^*$ then these equations can be rewritten in matrix form as:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{q}} \\ \dot{\hat{u}} \end{bmatrix} = \begin{bmatrix} -\varepsilon & -q^* & q^* \\ -(1-\sigma\alpha)\varepsilon & q^* & (1-\sigma\alpha)\varepsilon \\ 0 & -q^* & Bu^* \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{q} \\ \hat{u} \end{bmatrix} \tag{75}$$

where $\varepsilon = -\frac{(1-\alpha)}{\alpha}(B+\delta)^{1-\alpha} < 0$.

Let us denote the matrix as $\mathbf{D}$. Then we can compute the eigenvalues of the system from the characteristic matrix equation $\det(\lambda \boldsymbol{E} \quad \boldsymbol{D}) = 0$. Because $\det(\lambda \boldsymbol{E} \quad \boldsymbol{D}) = (\lambda - \varepsilon)(\lambda - q^*)(\lambda - Bu^*)$, the system has one negative and two positive eigenvalues, $\lambda_1 = \varepsilon < 0$, $\lambda_2 = q^* > 0$, and $\lambda_3 = Bu^* > 0$. Thus the system with two control and one state variable is saddle-path stable.

Solving the characteristic equations $(\lambda_i \mathbf{E} - \mathbf{D})\boldsymbol{\xi_i}\mathbf{=0}$ for $i = 1, 2, 3$ we get the eigenvectors $\boldsymbol{\xi_i}$ related to eigenvalues λ_i. Thus we get the following result

$$\boldsymbol{\Xi} = [\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, \boldsymbol{\xi_3}] = \begin{bmatrix} q^* - \sigma\alpha\varepsilon & q^* & 1 \\ (1-\sigma\alpha)\varepsilon & -\frac{1-\alpha}{\alpha}(\delta + B) & 0 \\ (1-\sigma\alpha)\varepsilon & q^* & 1 \end{bmatrix}. \tag{76}$$

If the vector of model variables is denoted as $\mathbf{Z} \equiv [x, q, u]^T$, then the system of log-linear differential equations can be written in the form $\dot{\hat{\mathbf{Z}}} = \mathbf{D}\hat{\mathbf{Z}}$ and the solution is given by

$$\hat{\mathbf{Z}}_t = e^{\mathbf{D}(t-t_0)}\hat{\mathbf{Z}}_{t_0} = \boldsymbol{\Xi} e^{\boldsymbol{\Lambda}(t-t_0)}\boldsymbol{\Xi}^{-1}\hat{\mathbf{Z}}_{t_0} \tag{77}$$

where $\boldsymbol{\Lambda} = diag[\lambda_1, \lambda_2, \lambda_3]$ and $\mathbf{D} = \boldsymbol{\Xi\Lambda\Xi^{-1}}$. Thus we can express the general solution in the form $\hat{\mathbf{Z}}_t = \sum_{i=1}^{3} \boldsymbol{\xi_i}\Theta_i e^{\lambda_i(t-t_0)}$ where $\boldsymbol{\Theta} = \boldsymbol{\Xi}^{-1}\hat{\mathbf{Z}}_{t_0}$.

Proposition 6 *(Stage of High Growth) The behavior of the economy given by (52)-(54) at the stage of high growth rate with $I = H$ can be approximated by the policy functions*

$$\tilde{q}_t - \tilde{q}_H^* = \kappa_H(\tilde{x}_t - \tilde{x}_H^*) \tag{78}$$

$$\tilde{u}_t - \tilde{u}_H^* = \kappa_H(\tilde{x}_t - \tilde{x}_H^*). \tag{79}$$

where the slope of policy functions is given by

$$\kappa_H = \frac{\xi_{H,12}}{\xi_{H,11}} = \frac{\xi_{H,13}}{\xi_{H,11}} = \frac{1 - \alpha\sigma\varepsilon_H}{q_H^* - \alpha\sigma\varepsilon_H} = \begin{cases} > 0, & \alpha\sigma > 1 \\ = 0, & \alpha\sigma = 1 \\ < 0, & \alpha\sigma < 1. \end{cases} \tag{80}$$

Proposition 7 *(Stage of Low Growth) The behavior of the economy given by (52)-(54) at the stage of high growth rate with $I = L$ can be approximated by*

$$\tilde{x}_t = \sum_{i=1}^{3} \xi_{L,i1} \Theta_{L,i} e^{\lambda_{L,i} t} + \tilde{x}_L^* \tag{81}$$

$$\tilde{q}_t = \sum_{i=1}^{3} \xi_{L,i2} \Theta_{L,i} e^{\lambda_{L,i} t} + \tilde{q}_L^* \tag{82}$$

$$\tilde{u}_t = \xi_{L,13} \xi_{L,11}^{-1} (\tilde{x}_t - \tilde{x}_L^*) + \tilde{u}_L^* \tag{83}$$

with $\mathbf{\Xi}_L$ and $\mathbf{\Theta}_L = \mathbf{\Xi}_L^{-1} \hat{Z}_{L,0}$ given by (76). However, for a level of human capital not too close to $\hat{H}$, the saddle path related to the BGP with low growth g_L^* is a good approximation:

$$\tilde{q}_t - \tilde{q}_L^* = \kappa_L (\tilde{x}_t - \tilde{x}_L^*) \tag{84}$$

$$\tilde{u}_t - \tilde{u}_L^* = \kappa_L (\tilde{x}_t - \tilde{x}_L^*). \tag{85}$$

the slopes of the policy functions are given by

$$\kappa_L = \frac{\xi_{L,12}}{\xi_{L,11}} = \frac{\xi_{L,13}}{\xi_{L,11}} = \frac{1 - \alpha\sigma\varepsilon_L}{q_L^* - \alpha\sigma\varepsilon_L} = \begin{cases} > 0, & \alpha\sigma > 1 \\ = 0, & \alpha\sigma = 1 \\ < 0, & \alpha\sigma < 1. \end{cases} \tag{86}$$

Appendix B: The Calibrated Model

In Section VI. we use numerical simulations to obtain precise interior solutions. Firstly, we specify the smooth function of the "learning curve" which captures the spillover effects of human capital accumulation as a logistic (or S-) curve

$$B(h) = \frac{B_H - B_L}{1 + e^{-\epsilon B_H (h - \hat{H})}} + B_L \tag{87}$$

where ϵ and $\hat{H}$ are parameters controlling the steepness and the position of the inflexion point.

To calibrate the model for US data we use similar parameter values as Lucas (1988) and Mulligan and Sala-i-Martin (1993): the intertemporal elasticity of substitution $\sigma = 0.5$, the capital income share $\alpha = 0.3$, the depreciation rate $\delta = .08$, and the rate of time preference $\rho = 0.08$. We further calibrate the model in such a way that the low stage growth g_L is equal to the average growth rate 2.8% during the 1950s and 1970s. We also extend our analysis in the sense that we are facing a new area of increasing returns related to the increased social effects of human capital accumulation. Our model has an interesting application for countries entering this new area, of which the United States is a leading example. We hypothesize that in the 1980s the USA was on the verge of a higher growth stage g_H^*, let's say, 4%. Thus the values of the efficiency of education are $B_L = 0.136$ and $B_H = 0.24$. The parameters of the learning curve were chosen as $\epsilon = 8$ and $\hat{H} = 8$. By means of equations (56)-(58) we can calculate steady state values $q_L^* = (c/k)_L^* = 0.612$, $x_L^* = (k/h)_L^* = 1.2697$, and $u_L^* = 0.7941$ for the lower growth stage and values $q_H^* = (c/k)_H^* = 0.680$, $x_H^* = (k/h)_H^* = 1.0316$, and $u_H^* = 0.75$ for the higher one. The results of simulation are shown in Figure X and Figure XI.

Appendix C: Proofs

Proof of Proposition 1 If we combine the first order conditions (34) and (38) $\lambda_{I,T_-} = c_{T_-}^{-\theta} e^{-\rho T} = \frac{\partial V_{II}(k_{T_-}, h_{T_-})}{\partial k_{T_-}} e^{-\rho T}$ we get $c_{T_-}^{-\theta} = \frac{\partial V_{II}(k_{T_-}, h_{T_-})}{\partial k_{T_-}}$. From dynamic programming we know that the partial derivation of a value function with respect to a state variable is equal to the shadow value of the state variable. Thus in our case $\frac{\partial V_{II}(k_{T_+}, h_{T_+})}{\partial k_{T_+}} = \lambda_{II,T_+}$. If we use this with the first order condition (39) we obtain $\lambda_{II,T_+} = c_{T_+}^{\theta} e^{-\rho(T-T)} = \frac{\partial V_{II}(k_{T_+}, h_{T_+})}{\partial k_{T_+}}$. This is equal to $\frac{\partial V_{II}(k_T, h_T)}{\partial k_T}$ if we take into account that for capitals $k_{T_-} = k_{T_+} = k_T$ and $h_{T_-} = h_{T_+} = h_T$ holds. Hence $\lambda_{I,T_-} = \lambda_{II,T_+} e^{-\rho T}$ and $c_{T_-} = c_{T_+}$. This means that the only variable which will adjust before the jump in productivity is consumption. The trajectory will therefore deviate from the saddle path only in q direction and u will move along the saddle path while after the increase in productivity it will jump to the high growth saddle path which means $u_{T_-} > u_{T_+}$. Q.E.D.

Proof of Proposition 4 If $h(0) < \hat{H}$ then the economy develops around the steady state L related to the low productivity B_L. Let $\tilde{x}_C$ be defined as the intersection of the saddle path $U(\tilde{x}; B_L)$ and the line $\tilde{u} = 0$. Assume now that the economy will never reach the critical level of human capital and thus the take-off never appears. If this is true, the economy would move along the stable saddle path $U(x; B_L)$. However, if the initial conditions are such that $\tilde{x}_0 \leq \tilde{x}_C$, the constraint is binding, implying that human capital does not grow and our assumption of no take-off is fulfilled. Thus the economy moves along the line $\tilde{u} = 0$ to the steady state with zero growth rate. Q.E.D.

Proof of Proposition 5 The solution for the time devoted to work before the productivity jump is determined by the low productivity saddle path $U(\tilde{x}; B_L)$. For given initial conditions $k(0)$ and $h(0)$, satisfying $\tilde{x}(0) = ln(k(0)/h(0)) > \tilde{x}_C$, any solution will be consistent with a take-off and convergence to the high growth stage only if the level of human capital attains the critical amount $h(t) = \hat{H}$ before $\tilde{x}(t) = \tilde{x}_C$. Otherwise there will eventually be a zero growth rate in human capital and the economy will not exhibit any adjustment to take-off and will move along the saddle path $U(x; B_L)$ from the beginning. Thus there must be a minimal level of human capital $H_C(\tilde{x}(0)) < \hat{H}$, which depends on $\tilde{x}(0)$ such that the economy will converge to the underdevelopment trap if $h(0) < H_C(\tilde{x}(0))$ and move to the take-off if $h(0) \geq H_C(\tilde{x}(0))$ where

$H_C(\tilde{x}(0)) = inf\{h(0) : h(T) < \hat{H}$ and $\tilde{x}T = \tilde{x}_C\}$. Q.E.D.

Proof of Proposition 6 The economy behavior given by (52)-(54) can be approximated by matrix equation (63) with the general solution given by (77) where $I = H$. Using the transversality conditions (22) we exclude explosive solutions related to positive roots and thus $\Theta_{H,2} = \Theta_{H,3} = 0$, and the solutions are

$$\hat{x}_H = \xi_{H,11}\Theta_{H,1}e^{\lambda_{H,1}t} \tag{88}$$

$$\hat{q}_H = \xi_{H,12}\Theta_{H,1}e^{\lambda_{H,1}t} \tag{89}$$

$$\hat{u}_H = \xi_{H,13}\Theta_{H,1}e^{\lambda_{H,1}t}, \tag{90}$$

where $\boldsymbol{\xi_{H,1}} = [\xi_{H,11}, \xi_{H,12}, \xi_{H,13}]^T$, and the policy functions can be expressed as

$$\tilde{q}_t - \tilde{q}_H^* = \frac{\xi_{H,12}}{\xi_{H,11}}(\tilde{x}_t - \tilde{x}_H^*) \tag{91}$$

$$\tilde{u}_t - \tilde{u}_H^* = \frac{\xi_{H,13}}{\xi_{H,11}}(\tilde{x}_t - \tilde{x}_H^*). \tag{92}$$

After substitution from (76) the slopes of policy functions are identical and given by

$$\kappa_H = \frac{\xi_{H,12}}{\xi_{H,11}} = \frac{\xi_{H,13}}{\xi_{H,11}} = \frac{1 - \alpha\sigma\varepsilon_H}{q_H^* - \alpha\sigma\varepsilon_H} = \begin{cases} > 0, & \alpha\sigma > 1 \\ = 0, & \alpha\sigma = 1 \\ < 0, & \alpha\sigma < 1 \end{cases} \tag{93}$$

Q.E.D.

Proof of Proposition 7 The solution[2] of the log-linear approximation of the model is given by (77) together with the initial and connecting conditions from (49) and (50) from Proposition 1, implying continuity in consumption and no adjustment in working time before the miracle occurs. Thus the solution is given by the following conditions

$$\tilde{x}_t = \sum_{i=1}^{3} \xi_{L,i1}\Theta_{L,i}e^{\lambda_{L,i}t} + \tilde{x}_L^* \tag{94}$$

$$\tilde{q}_t = \sum_{i=1}^{3} \xi_{L,i2}\Theta_{L,i}e^{\lambda_{L,i}t} + \tilde{q}_L^* \tag{95}$$

$$\tilde{u}_t = \xi_{L,13}\xi_{L,11}^{-1}(\tilde{x}_t - \tilde{x}_L^*) + \tilde{u}_L^* \tag{96}$$

[2]To be able to specify the solution of the model we need to determine the initial condition for q and the time of the productivity jump T. This has been solved in a working paper version of this paper [Kejak, 1998]. It is not presented here due to lack of space. However, the qualitative analysis does not depend on it.

with $\boldsymbol{\Xi}_L$ and $\boldsymbol{\Theta}_L = \boldsymbol{\Xi}_L^{-1}\hat{Z}_{L,0}$ given by (76). It follows directly from the Turnpike Theorem property of the time paths of model variables that the behavior of the economy with a level of human capital not too close to $\hat{H}$ can be well approximated by the saddle path related to the BGP with low growth g_L^*. Thus the low growth rate stage can be described analogously to (78)-(80) for these values of human capital. Q.E.D.

Lemma 1 *Any solution of the model given by (63) and (49)-(51) must satisfy the following four conditions*

$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{q}_0 \end{bmatrix} = \boldsymbol{W}\left\{ e^{-\boldsymbol{D}_L T}\left(\begin{bmatrix} \tilde{x}_H^* \\ \tilde{q}_H^* \\ \tilde{u}_H^* \end{bmatrix} + \begin{bmatrix} \xi_{H,11} \\ \xi_{H,12} \\ \xi_{H,13} \end{bmatrix} \xi_{H,11}^{-1}(\tilde{x}_T - \tilde{x}_H^*) \right) - \left(\boldsymbol{E} - e^{-\boldsymbol{D}_L T} \right) \begin{bmatrix} \tilde{x}_L^* \\ \tilde{q}_L^* \\ \tilde{u}_L^* \end{bmatrix} \right\} \tag{97}$$

$$\tilde{u}_0 = \frac{\xi_{L,13}}{\xi_{L,11}}(\tilde{x}_0 - \tilde{x}_L^*) + \tilde{u}_L^* \tag{98}$$

and

$$h_T = h_0 e^{\gamma_L^* T} e^{\Gamma_T} = \hat{H} \tag{99}$$

with

$$\Gamma_T = \frac{u_L^*}{1 - u_L^*}\gamma_L^* \frac{\xi_{L,13}}{\xi_{L,11}} \sum_{i=1}^{3} \xi_{L,i1}\Theta_{L,i}\lambda_{L,i}^{-1}\left(1 - e^{\lambda_{L,i}T}\right) \tag{100}$$

where $\gamma_t = \dot{h}_t/h_t$, $\gamma_L^* = B_L(1-u_L^*)$ *and* $\boldsymbol{W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *which create the system of four equations in four unknowns* $\tilde{x}_T$, $\tilde{q}_0$, $\tilde{u}_0$ *and* T.

Proof of Lemma 1 We know that at date T_+, after the jump, the economy must be on its stable saddle path given by (88)-(90) i.e.

$$\tilde{\mathbf{Z}}_{T_+} - \tilde{\mathbf{Z}}_H^* = \begin{bmatrix} \tilde{x} - \tilde{x}_H^* \\ \tilde{q} - \tilde{q}_H^* \\ \tilde{u} - \tilde{u}_H^* \end{bmatrix} = \boldsymbol{\xi}_{H,1}\xi_{H,11}^{-1}(\tilde{x}_{T_+} - \tilde{x}_H^*). \tag{101}$$

The solution at date T_-, before the jump, is given by $\tilde{\mathbf{Z}}_{T_-} - \tilde{\mathbf{Z}}_L^* = e^{\mathbf{D}_L T}(\tilde{\mathbf{Z}}_0 - \tilde{\mathbf{Z}}_L^*)$. From Proposition 1 we know that the only variable which jumps at time T is the time devoted to work and the size of the jump is given by the switch between low growth saddle path and that of high growth. Using this result together with equation (101) we get

$$\tilde{\mathbf{Z}}_{T_+} = \boldsymbol{\xi}_{H,1}\xi_{H,11}^{-1}(\tilde{x}_T - \tilde{x}_H^*) + \tilde{\mathbf{Z}}_H^* = e^{\mathbf{D}_L T}(\tilde{\mathbf{Z}}_0 - \tilde{\mathbf{Z}}_L^*) + \tilde{\mathbf{Z}}_H^*. \tag{102}$$

Because we know that the initial condition for u is simply given by $\tilde{u}_0 = \frac{\xi_{L,13}}{\xi_{L,11}}(\tilde{x}_0 - \tilde{x}_L^*) + \tilde{u}_L^*$, we obtain the equation for the initial conditions for x and q as

$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{q}_0 \end{bmatrix} = \boldsymbol{W} \left\{ e^{-\boldsymbol{D}_L T} \left(\begin{bmatrix} \tilde{x}_H^* \\ \tilde{q}_H^* \\ \tilde{u}_H^* \end{bmatrix} + \begin{bmatrix} \xi_{H,11} \\ \xi_{H,12} \\ \xi_{H,13} \end{bmatrix} \xi_{H,11}^{-1}(\tilde{x}_T - \tilde{x}_H^*) \right) - \left(\boldsymbol{E} - e^{-\boldsymbol{D}_L T} \right) \begin{bmatrix} \tilde{x}_L^* \\ \tilde{q}_L^* \\ \tilde{u}_L^* \end{bmatrix} \right\}. \tag{103}$$

We know that $h_T = h_0 e^{\int_0^T \gamma_t dt}$ where the growth rate of human capital is $\gamma_t = \dot{h}_t / h_t = B_l(1 - u_t)$. If we approximate the growth rate $\gamma_L^* d \ln \gamma_t \approx -B_L u_L^* d \ln u_t$ and $\gamma_L^* = B_L(1 - u_L^*)$ then $\hat{\gamma}_{L,t} = -\frac{u_L^*}{1-u_L^*}\hat{u}_{L,t}$ and $\gamma_t = \gamma_L^*(1 + \hat{\gamma}_{L,t})$. Thus $h_T = h_0 e^{\gamma_L^* T} e^{\Gamma_T}$ where Γ_T can be derived in the following way: $\Gamma_T = \gamma_L^* \int_0^T (\tilde{\gamma}_t - \tilde{\gamma}_L^*) dt = \gamma_L^* \int_0^T (-\frac{u_L^*}{1-u_L^*})(\tilde{u}_t - \tilde{u}_L^*) dt$. When we apply the formula for the solution of $\tilde{u}_t$, before time T the solution does lie on the saddle path, we get $-\frac{u_L^*}{1-u_L^*}\gamma_L^* \int_0^T \frac{\xi_{L,13}}{\xi_{L,11}}(\tilde{x}_0 - \tilde{x}_L^*) dt$. Substituting for the solution of x we obtain $-\frac{u_L^*}{1-u_L^*}\frac{\xi_{L,13}}{\xi_{L,11}}\gamma_L^* \int_0^T \sum_{i=1}^3 \xi_{L,i1}\Theta_{L,I} e^{\lambda_{L,i} t} dt$. Solving the integral we obtain

$$\Gamma_T = \frac{u_L^*}{1 - u_L^*} \frac{\xi_{L,13}}{\xi_{L,11}} \gamma_L^* \sum_{i=1}^3 \xi_{L,i1} \Theta_{L,i} \lambda_{L,i}^{-1} (1 - e^{\lambda_{L,i} T}). \tag{104}$$

Q.E.D.

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1. In the original paper by Lucas the external effect was in the goods sector.
2. In contrast to the authors, who use the overlapping generation framework, we use the infinite lifetimes one.
3. Recently, the existence of indeterminacies in Lucas' model has been discovered and published in Benhabib and Perli [1994] and Xie [1994]. We will briefly discuss this problem later and suggest how this kind of indeterminacies may be removed.
4. When economies are subjected to structural changes, the new steady states may be different from the original ones (see, for instance, the case of the post-socialist CEE countries).
5. In the sense of Lucas by theory we mean '..an explicit dynamic system, something that can be put on a computer and run' [Lucas, 1988].
6. This is different from the standard learning-by-doing models [Arrow, 1962; Romer, 1986] and from Zilibotti [1995] where technical progress is a by-product of the investment in physical capital.
7. Recently Benhabib and Spiegel [1994] have provided the empirical evidence confirming that per capita income growth depends positively on average levels of human capital.
8. This specification was used in Lucas [1988].
9. Whenever possible we suppress time indices to avoid cluttered notation. A dot denotes a time derivative.
10. Note that w denotes the wage rate per efficiency unit i.e. an agent with human capital h working u fraction of his time endowment earns labor income $wl = wuh$.
11. The necessity of qualitative approximations in obtaining insights into human reasoning as well as in structuring results in modeling complex systems are well known in the field of artificial intelligence and physics (See for example Kuipers [1986] and Lum and Chua [1991].)
12. In Kejak [1998] the 'qualitative' approximation is accompanied by the 'quantitative' one, a piece-wise log-linearization around particular steady states.
13. We can look at the approximation of the logistic curve as a piece-wise linearization.
14. The same transformation is used in Mulligan and Sala-i-martin [1993] and Benhabib and Perli [1994].
15. The presence of globally increasing returns can also create another kind of problem as in the original Lucas [1988] model. In that model human capital externalities in the goods production can (for realistic values of the externality factor) cause a "continuum" type of indeterminacy with the implication that the model can exhibit multiple growth rates for given physical and human capital endowment (see Benhabib and Perli, 1994; and Xie, 1994). This rather "strange" property could be "cured" by applying upper-bounded increasing returns as is done here.
16. See Mulligan and Sala-i-martin [1993].

17. The situation when people spend all their time in schools and do not work seemingly implies the maximum growth in human capital. This is not feasible, however, on the BGP because the output F in equation (17) is zero and physical capital is, therefore, consumed and steadily declines rather than growing at the rate $g_I^* > 0$. This implies that $u_I^* > 0$.

18. Because the eigenvalues of the system do not change their signs for any feasible values of parameters there are no indeterminacies in the model.

19. The policy functions do not depend on x if $\alpha = \sigma^{-1}$.

20. For the analysis of transitional dynamics the time elimination method has been used in Mulligan and Sala-i-martin [1993] and the projection method (introduced for the application in economics by Judd [1992]) in Kejak [1993].

21. As we already mentioned above we will confine our analysis only to this case here for two reasons. First, it seems that the chosen case is more empirically relevant and second, the analysis of the other case is completely symmetrical to our case.

22. However, there is already some adjustment in the economy before it reaches the critical value. Because people know all future prices perfectly, they have sufficient time to adjust to the productivity miracle and smooth their consumption stream.

23. We allow parameter ρ to take values only from the interval $(\frac{(1-\sigma)\sigma}{(1-\sigma)\sigma+1}B_H, B_H)$ where the lower boundary determines the highest achievable BGP with a bounded lifetime utility and the upper boundary corresponds to the BGP with zero growth related to high productivity B_H.

24. The latter is not really an explanation because the decline in the productivity is assumed to be exogenous.

25. The same argument related to the revolution in information technologies has been used in Greenwood and Yorukoglu [1997].

26. This is consistent with the empirical observations presented in Bailey and Schultze [1990] which show that the productivity slowdown is accompanied with a decline in the rate of growth of the capital stock.