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Revealing Private Information in a Patent Race*

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Abstract

In this paper I investigate the role of private information in a patent race. Since firms often do their research in secrecy, the common assumption in patent race literature that firms know each other's position in the race is questionable. I analyze how the dynamics of the game changes when a firm's progress is its private information, and I address the question whether revealing it might be to a firm's advantage. I find that a firm has an incentive to reveal its breakthrough only if its rival has not done so, and only if the research is costly.

Keywords: Patent Race, R&D Investment, Race, Optimal Effort, Revealing Private Information.

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1 Introduction

Patent race has been studied extensively since the seminal paper of Loury (1979). Many studies analyze how firms' R&D investments vary as firms' positions in the race evolve. However, that type of analysis assumes that firms know each other's progress at any point in time. This study is devoted to lifting this extreme assumption and investigating whether firms have the incentive to reveal their progress voluntarily.

Dropping the assumption of complete information in a patent race entirely changes the dynamics of the game. Whilst in a race with complete information the state of the game evolves in discrete steps, in a race with private information each player continuously updates his posterior belief about the rival. As a result, unlike in the patent race with complete information, firms continuously adjust the intensity of their R&D investments. How then, does a firm's R&D effort evolve? Do firms invest most intensely early on in the game or do they invest increasingly aggressively over time? How does a firm's R&D effort change with the arrival of a breakthrough? And most importantly, can a firm discourage its rival by revealing its progress towards a patent?

Although this study focuses on the patent race, it can be seen as an example of a broader class of races. In their recent paper, Seel and Strack (2016) point out that, the majority of the race, contest, and tournament literature is such that either the game is static or it is dynamic with publicly observed states, while there is very little known about the games in which each player's state is observed privately. Philipp Strack has coauthored several studies exploring the role of private information in games in which each player's only action is the choice of a stopping time. In this study, I investigate the role of private information in games in which players have the choice of effort at any time. Whilst the optimal stopping is the irreversible decision to stop a certain process once and for all, the choice of effort level allows players to adapt their level of participation to the momentary situation. Finally, it is natural to ask which of the information settings (public and private) is better for the players, and what would happen if the players themselves had the control over the information setting by having the option to reveal their progress.

I study a private information version of the patent race introduced in Grossman and Shapiro (1987). There are two firms that compete in making a discovery. For each firm there is an interim state on the way to a patent that is reached after making the first breakthrough. Reaching the interim state is the firm's private information unless the firm discloses it. Throughout this study disclosure is always verifiable, not cheap talk.[1] Either there is no credible way for firms to disclose breakthroughs, and then we refer to the situation as a private information setting, or the breakthroughs can be disclosed verifiably without any risk of technological spillovers, which we call a private information setting with an option to reveal. In contrast, there is no credible way to reveal not having made a breakthrough. Each

[1] For example, a pharmaceutical firm can publish results of audited randomized trials, without publishing details about the drug. A technological firm can publish a video demonstrating the function of its prototype, as Samsung did recently with its folding smartphone. Another example could be the rocket launches of Space-X, which publicly demonstrate certain capabilities of their rockets without disclosing the technology behind.

firm continuously chooses any positive research effort level that translates into its hazard rate of making a breakthrough. The firm incurs a cost flow that is a quadratic function of its effort. The game ends when either firm completes both stages of R&D and wins the patent.

I consider four different information settings in order to address different types of questions. First, I solve the complete information version of the patent race and provide some intuition for what to expect in the private information setting. Next, I study the private information version of the patent race in which players cannot reveal their breakthroughs. Further, the asymmetric version of the private information setting is the case in which one of the two firms is known to be successful (having made the first breakthrough). Finally, I address the question whether firms would reveal their breakthroughs voluntarily if they had the option to reveal their breakthroughs in a verifiable way without leaking any technological secrets.

In the private information setting, a firm only observes its own progress, and it updates its belief about the rival. I show that the posterior probability that the rival has completed the first stage keeps increasing over time and it converges to a steady-state value that is strictly less than 1. On the one hand, the rival is increasingly likely to have completed the first stage of R&D over time. On the other hand, the longer time has passed since the rival completed the first stage of R&D, the more likely he would have patented already. These two factors drive the posterior belief about the rival's success. As a player becomes increasingly convinced that his rival has already made the first breakthrough, he becomes increasingly pessimistic about his own situation. I show that if a player still has not completed the first stage of R&D, the increasing pessimism makes him decrease effort over time. However, once a player completes the first stage of R&D, his effort increases and continues increasing until either of the players makes a patent. This is in line with standard findings in studies of the complete information versions of patent races, i.e., that players exert the highest effort when they are neck and neck.

Next, I study the asymmetric situation in which one player, say A, is known to be one step away from patenting, while the other player, B, starts two steps away from patenting. Such a situation is interesting on its own, but understanding it is essential to further analyze the players' decision to reveal breakthroughs. Since player A does not observe player B's state, he updates his posterior belief about player B's progress and adjusts his effort accordingly. As a result, player A increases his effort over time. In contrast, player B's belief does not change as he knows that his rival A is just one step away from patenting. Nevertheless, player B's effort changes continuously as he reacts to player A's increasing rivalry. How do the efforts of the two players compare if they are both one step away from patenting, and only their information about each other differs? I show that if both players are one step away from patenting, then player B makes higher effort than player A because he knows that the players are neck and neck. As a result, player A is worse off than player B.

Finally, I study the patent race with the option to disclose breakthroughs. My first result in this game is that a player never discloses a breakthrough knowing that his rival is successful. In fact, a successful player becomes only encouraged to work harder by learning about the rival's success. Thus, if one player has revealed, the other player will keep his progress secret forever. Knowing that, I analyze firms' strategies before any of them has revealed success. Such a situation is symmetric. Let us consider the incentives of player A. Revelation of player A's success has an ambiguous effect on his rival's (player B's) effort: The desirable effect of the revelation is that it discourages the rival if the rival is unsuccessful. However, there are two undesirable effects that come with the revelation. First, if the rival is already successful, or once he becomes successful, having the information about player A's success causes him to increase effort. Second, player A's revelation gives player A an informational disadvantage, as player B will be informed about player A's success and he will never reveal his own success afterwards. Evaluating these tradeoffs numerically, I conclude that a firm reveals instantly only if the research is difficult in the sense that making a breakthrough is a long-term project. If the research is difficult, the rival is expected to remain unsuccessful for a long time, and so the desirable effect of discouraging the unsuccessful rival dominates the two undesirable effects. In contrast, if the research is easy, then players never reveal as each of them expects the rival to catch up promptly. Finally, in the case of a moderate research difficulty, a successful player has a tendency to reveal, yet he prefers to wait for his rival to do so. Players randomize over their revelation decision and the equilibrium resembles the one known from the war of attrition games.

It is natural to ask which of the information settings is ex-ante better for the players. The way various information settings rank from the welfare perspective depends on the research difficulty. However, regardless of the research difficulty, it can be concluded that players do better under voluntary revelation than under no revelation (private information setting) or under mandatory revelation (complete information setting).

The structure of this paper is as follows: Section 2 introduces the model of patent race that is studied, and analyzes the complete information version of it as a benchmark. Section 3 analyzes a firm's R&D effort over time in the private information game (with no option to reveal). Section 4 analyzes the special case in which one of the players is known to be successful. Finally, using the insights from the previous sections, Section 5 studies the case in which players have the option to reveal their success. Criteria for the existence of no-revelation equilibrium and instant-revelation equilibrium are provided. Finally, the mixed-revelation equilibrium is characterized. Section 6 provides a welfare comparison and extensions. Section 7 concludes.

1.1 Literature Review

The nature of R&D investments under a competitive environment was first studied by Loury (1979); Lee and Wilde (1980); and Dasgupta and Stiglitz (1980). In the studies, the patent race is static in the sense that the firms choose their R&D effort once and for all at the beginning. Shortly after, the analysis was extended into a dynamic environment by Grossman and Shapiro (1987) and Harris and Vickers (1987). Their models feature a specific finishing line: a firm wins the patent race once it completes a given number of R&D stages. One of their main results is that the firms invest in R&D most intensively when they are equally close to the finishing line. This is in contrast with the finding of Hörner (2004), who studies a perpetual race in which the firm that is ahead of the other receives flow payoff. Hörner shows that in the case of the perpetual race the competition is not necessarily fiercest when the firms are closest. Both Judd (2003) and Moscarini and Smith (2007) study continuous state-space versions of Harris and Vickers (1987) and do normative analysis of the game, finding that innovators over-invest on risky projects, and that leaders invest more than followers. Anderson and Cabral (2007) study a dynamic race in which players choose between risky and safe technology. In all these studies, players' progress in the race is common knowledge.

The paper by Dosis et al. (2013) considers a patent race with two stages, called the research phase and the development phase. In either phase, the breakthroughs arrive in a random fashion, and in addition to that, in the research phase there is uncertainty about the innovation being feasible. One of their main findings is that under-investment is the dominant effect in the initial stage of the race (research phase), while over-investment is the dominant effect in the more advanced stages (development phase). Another recent study by Hopenhayn and Squintani (2016) investigates firms' allocation of R&D investments across research areas and concludes that firms overinvest in high return areas.

There are various studies of races featuring some type of private information about a player's success: In Akcigit and Liu (2015) firms have private information about dead-end projects, and they show that a firm is silent about abandoning a bad project in order to let its rival misallocate R&D investments. Private information about a breakthroughs is present in Hopenhayn and Squintani (2015), in which a firm makes a discovery at a random time, and its value subsequently grows during the developmental phase until the firm patents it; the firm faces a tradeoff between increasing the value of its potential patent and risking being pre-empted by its rival. A race with private information is also studied in Moscarini and Squintani (2010) and Hopenhayn and Squintani (2011), in which players choose optimal stopping instead of effort.

The closest related study is the working paper of Gordon (2011), who also studies a two stage patent race with private information, but firms' effort level is restricted to low and high only. Restricting the effort choice down to two effort levels dramatically simplifies the analysis of the model. However, the restriction has two crucial downsides: First, unlike in this study, the game might have no or multiple equilibria, depending on the choice of the parameters, as a result of the effort choice being discrete. Second, the revelation has an impact only in the rare situation that it flips which of the two effort levels is optimal. In contrast, in this paper players adapt their effort level to any minor changes in their incentives.

Although Bonatti and Hörner (2011) study collaboration in place of competition, it shares some features with this study. There are two players collectively engaged in a project. The players dynamically choose their effort that determines the hazard rate of completion of the project. There is uncertainty about the quality of the project and over time the players become increasingly pessimistic about the project being good. Most of their work assumes that effort cost is linear, which allows for the use of the bang-bang solution method and simplifies the analysis. In contrast, in this paper we cannot use the bang-bang solution method and thus a completely different solution technique is needed.

There is a broad literature on disclosure. Jovanovic (1982) studies disclosure of product quality, and, in contrast to Akerlof (1970), he assumes that truthful disclosure is feasible.

Jovanovich finds that competition in the free market drives the amounts of disclosure beyond the socially-optimal level. Dye (1985) and Milgrom (2008) study a firm's disclosure of information as a form of persuasion of potential investors. An extensive overview of literature on quality disclosure can be found in Dranove and Jin (2010). In all of the above literature the firms disclosure is studied in relation to a consumer or an investor. In contrast, in this study the strategic disclosure occurs between two rivals.

Studies of disclosure in a patent race include the following literature. Both Lichtman et al. (2000) and Baker and Mezzetti (2005) study disclosure as a way to increase the prior art in order to prevent or delay the rival from patenting a new technology. Gill (2008) studies a firm's tradeoff between disclosing its progress in order to discourage the rival from investments and the potential technological spillover. Aoki and Spiegel (2009) study the impacts of the pre-grant publications of patents that are mandatory within 18 month of patent application in most industrial countries except for the US. The study by Kultti et al. (2007) investigates whether secrecy or patenting is a better way of protecting intellectual property, when being concerned about the technological spillovers.

2 Model

I study an infinite horizon continuous time model of a patent race with two risk-neutral firms A and B. The firms invest in R&D with the aim to win a specific patent. It takes two consecutive breakthroughs to be able to patent. Define the state $x_t^j \in \{0, 1, 2\}$ of the firm $j \in \{A, B\}$ as the number of breakthroughs the firm has made by time t. Initially, both firms start in the state 0; once a firm makes the first breakthrough its state is 1; and when making the second breakthrough, the firm patents (state 2) and wins the value patent of the patent $v > 0$. We assume that each realization of the trajectory $t \mapsto x_t^j$ is right-continuous.

2.1 Actions and Payoffs

2.1.1 Effort

At any time $t \geq 0$, each firm $j \in \{A, B\}$ chooses its research effort $e_t^j \geq 0$ which is defined as the hazard rate of making a discovery. Note that this means that the knowledge is not accumulated unless a breakthrough is made – the research consists of independent trials. We require every realization of the trajectory $t \mapsto e_t^j$ to be a right-continuous function. Then[2]

$$P[x_{t+\Delta t}^j = x_t^j + 1] = e_t^j \Delta t + o(\Delta t).$$

The research effort is a result of R&D investments, and thus it is costly. Player j incurs flow cost $c(e_t^j)$ that is a function only of the current effort. The marginal cost of effort is increasing; in particular, we assume that the cost function has the quadratic form $c(e) = \frac{1}{2}\alpha e^2, \alpha > 0$.

[2] $o(\cdot)$ represents any function such that $o(\Delta t)/\Delta t \to 0$ as $\Delta t \searrow 0$

2.1.2 Expected utility

Future payoffs are discounted at rate $r > 0$. The firm that patents first receives the prize v. Both firms have to bear their research flow costs accumulated over time. The expected utility of player j then is

$$EU^j = \mathbb{E}\left[\underbrace{-\int_0^{\tau} \exp(-rt) \cdot c(e_t^j) dt}_{\text{effort cost}} + \underbrace{\exp(-r\tau) \cdot v \cdot \mathbf{1}_{\tau=\tau^j}}_{\text{value of the patent}} \right],$$

where τ^j is the time at which player j files a patent (or infinity) and $\tau = \min\{\tau^A, \tau^B\}$.[3]

2.2 Complete Information Case

As a benchmark, let us first look at the version of the patent race in which players' progress is common knowledge, such that each player knows the state his rival is in.[4]

The state of the game is given by the combination of the state $y, z \in \{0, 1, 2\}$ of player A and B, respectively. Restrict our attention to symmetric Markov perfect equilibria.[5] Denote v^{yz} and e^{yz} player A's continuation value and effort, respectively. The continuation value and effort of player B then is v^{zy} and e^{zy}, respectively.

The player's continuation value can be characterized recursively as

$$v^{yz} = \max_{e \geq 0} \left\{ v^{y+1,z} e\Delta t - \frac{\alpha}{2} e^2 \Delta t + v^{y,z+1} e^{zy} \Delta t + (1 - (e + e^{zy})\Delta t)(1 - r\Delta t) v^{yz} + o(\Delta t) \right\}.$$

Subtracting v^{yz} from both sides of the equation, dividing by $\Delta t > 0$, and letting $\Delta t > 0$ to zero, we obtain

$$0 = \max_{e \geq 0} \left\{ v^{y+1,z} e - \frac{\alpha}{2} e^2 + v^{y,z+1} e^{zy} - (e + e^{zy} + r) v^{yz} \right\}.$$

The first order condition yields that the optimal effort e is equal to $e^{yz} = \frac{1}{\alpha}(v^{y+1,z} - v^{yz})$, which means that it is proportional to the potential gain from making a breakthrough. The continuation value is then given by the system of equations

$$0 = \frac{\alpha}{2}(e^{yz})^2 - e^{zy}(v^{yz} - v^{y,z+1}) - rv^{yz}, \quad y, z \in \{0, 1\}$$

along with the boundary conditions $v^{2,z} = v$ and $v^{y,2} = 0$.

[3] The function $\mathbf{1}_{\tau=\tau^j}$ is 1 if player j wins the patent, and 0 otherwise. (The case of both players patenting simultaneously can be neglected as it occurs with zero probability.)

[4] This model has been analyzed in Harris and Vickers (1987) under the assumption $r = 0$, or in Grossman and Shapiro (1987), in which a more general class of cost function is considered, and hence, a less clear conclusion can be made.

[5] In fact, the game has only the symmetric equilibrium, as can be shown by making a minor modification to the proof of uniqueness of symmetric equilibrium. We assume symmetry here for the sake of notation simplicity.

This system of equations has a unique solution, which allows us to compare players' research efforts among different scenarios (the proof of the following proposition can be found in the Appendix).[6]

Proposition 1. *The patent race with complete information has a unique symmetric Markov perfect equilibrium, in which*

(i) $e^{01} < e^{00}$ *... an unsuccessful player is discouraged from exerting effort by the success of its rival;*

(ii) $e^{10} < e^{11}$ *... a successful player is encouraged to exert effort by the success of its rival.*

The takeaway from these observations is that being successful (in state 1) discourages a rival that is unsuccessful ($e^{01} < e^{00}$) from exerting effort, but it encourages a rival that is successful ($e^{10} < e^{11}$). Thus, we can expect that in the private information setting in which a player does not observe a rival's state and only updates the posterior belief about him, an unsuccessful player will continuously decrease effort as a result of increasingly being convinced about the rival having a lead. Conversely, a successful player will continue to increase effort over time as he is increasingly likely to be in the neck and neck situation.

The above results also shed some light on the question whether players want to reveal their success when having the option to do so credibly without leaking any technological secrets. If the above observations from the public information setting are valid even in the setting with the option to reveal success, then we could deduce that a player would reveal success only when expecting his rival is likely to still be unsuccessful and not to catch up any soon.

2.3 Private Information Case

I proceed with modeling the private information about firms' progress in the patent race.

Firms often do not observe each other's research progress on the way to a patent. In this section, I assume that the state x_t^j of firm j is its private information, thus each firm knows its own progress towards the patent, but it does not observe the progress of its rival. In addition, I assume that the effort e_t^j is not observed by the rival, otherwise the rival might be able to infer the state x_t^j from the effort e_t^j.

Given that a player does not know its rival's state, he infers a posterior belief about it. Define p_t^j as player $-j$'s posterior belief about player j being in state 1:

$$p_t^j = P[x_t^j = 1 \mid x_t^j < 2].$$

Note that the probability of $x_t^j = 1$ is conditioned on $x_t^j < 2$, because the only information player $-j$ has about it is that j has not patented yet.

We are interested in Markov perfect equilibria (MPE) in the game. Thus, players condition their actions on the payoff relevant state, which consists of his state x_t^j and the profile

[6] Grossman and Shapiro (1987) showed that $e^{11} > e^{10} > e^{01}$. However, since they allow for a more general class of convex effort cost functions, they conclude that any relationship between e^{00}, e^{01} and e^{10} is possible.

of mutual posterior beliefs (p_t^j, p_t^{-j}). Player j's strategy is then the choice of the effort $e \geq 0$ as a function of the profile (x_t^j, p_t^j, p_t^{-j}). To simplify the notation, we will denote player j's effort when in state $x \in \{0,1\}$ at time t as e_t^{xj}, since the profile (p_t^j, p_t^{-j}) is only a function of the time. However, we need to keep in mind that $e_{t_1}^{xj}$ has to be identical to $e_{t_2}^{xj}$, whenever $(p_{t_1}^j, p_{t_1}^{-j}) = (p_{t_2}^j, p_{t_2}^{-j})$. Later, when we restrict our attention to symmetric MPEs, we will use the notation $e^x(p)$ to represent a player's effort when in state $x \in \{0,1\}$ and when $p_t^j = p_t^{-j} = p$.

2.3.1 Continuation Value

Every state has an associated continuation value v; we will use the same system of notation for the continuation value as for the effort. The continuation value of player j in state x at time t is

$$v_t^{xj} = \max_{e \geq 0} \left\{ v_t^{x+1,j} e\Delta t - \tfrac{\alpha}{2}(e)^2 \Delta t + [1 - (r + e + f_t^{-j})\Delta t] v_{t+\Delta t}^{xj} + o(\Delta t) \right\},$$

where f_t^{-j} is the hazard rate with which the rival patents at time t. Subtracting v_t^{xj} from both sides of the equation, dividing them by Δt, and letting Δt to zero from above, we obtain

$$-\dot{v}_t^{x,j} = \max_{e \geq 0} \left\{ (v_t^{x+1,j} - v_t^{x,j})e - \tfrac{\alpha}{2}(e)^2 - (r + \phi_t^{-j}) v_t^{x,j} \right\}.$$

The first order condition for e implies $e_t^{x,j} = \frac{1}{\alpha}(v_t^{x+1,j} - v_t^{x,j})$, in other words, player j's optimal effort is equal to the potential gain from instant completion of the current stage of R&D. We obtain the following differential equation for the value function

$$-\dot{v}_t^{x,j} = \tfrac{\alpha}{2}(e_t^{x,j})^2 - (r + \phi_t^{-j}) v_t^{x,j}.$$

2.3.2 Posterior Belief

At time $t = 0$ the posterior belief is $p_0^j = 0$ as players start from the state 0 with certainty. Using Bayes' law, we obtain the following law of motion for the posterior belief.

Lemma 1. *The law of motion for p_t^j is $\dot{p}_t^j = (1 - p_t^j)(e_t^{0j} - p_t^j e_t^{1j})$.*

One might expect that p_t^j will eventually approach 1, but this is not the case. Although the rival is increasingly likely have made a breakthrough, he is also increasingly likely to have patented already. Thus, conditioned on the fact that the rival has not patented, his probability of being in state 1 asymptotically converges to a value below 1.

It remains to express the hazard rate ϕ_t^{-j} with which the rival files the patent at time t. The rival has made the first breakthrough with probability p_t^{-j}, and if that is the case, then he patents with the hazard rate $e_t^{1,-j}$. Accordingly, $f_t^{-j} = p_t^{-j} e_t^{1,-j}$.

The game can be summarized by the following system of ODEs:

$$\begin{aligned} -\dot{v}_t^{1j} &= \tfrac{\alpha}{2}(e_t^{1j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{1j} \\ -\dot{v}_t^{0j} &= \tfrac{\alpha}{2}(e_t^{0j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{0j} \\ \dot{p}_t^j &= (1 - p_t^j)(e_t^{0j} - p_t^j e_t^{1j}). \end{aligned} \tag{1}$$

2.3.3 Normalization of Parameters

The presented model of the patent race includes three parameters: the value of the patent v, the effort cost multiplier α, and the discount rate r. However, the generality of the problem will not be compromised if we set $v = 1$ and $\alpha = 1$.

Proposition 2. *Any equilibrium in the patent race with private information corresponds to an equilibrium of the game with $\hat{v} = 1$, $\hat{\alpha} = 1$, and $\hat{r} = \frac{\alpha r}{v}$.*

The intuition behind this result is that, apart from choosing a unit of value such that $v = 1$, it is also possible to choose the units of time such that $\alpha = 1$.

The normalization allows us to simplify the notation of the proofs. It also has the advantage that if some property can only be shown numerically, then the property can be tested in a one-dimensional parameter space.

3 Patent Race with Private Information

In this section, I show existence of unique Symmetric Markov perfect equilibrium, and I study its properties such as the evolution of a firm's R&D effort over time.

3.1 Unique Equilibrium

The object of our interest is a symmetric Markov perfect equilibrium.[7] The symmetry reduces the system of six ODEs (1) into the following system of three ODEs:

$$\begin{aligned} -\dot{v}_t^1 &= \tfrac{\alpha}{2}(e_t^1)^2 - (r + p_t e_t^1)v_t^1 \\ -\dot{v}_t^0 &= \tfrac{\alpha}{2}(e_t^0)^2 - (r + p_t e_t^1)v_t^0 \\ \dot{p}_t &= (1 - p_t)(e_t^0 - p_t e_t^1), \end{aligned} \tag{2}$$

where $e_t^1 = \frac{1}{\alpha}(v - v_t^1), e_t^0 = \frac{1}{\alpha}(v_t^1 - v_t^0)$, together with the initial condition $p_0 = \hat{p}$, and inequalities $0 \le v_t^0 \le v_t^1 < v$ and $p_t \in [0, 1]$, for all $t \ge 0$.

To satisfy the Markov condition, it has to be the case that for $x \in \{0, 1\}$ we have $e_{t_1}^x = e_{t_2}^x$ whenever $p_{t_1} = p_{t_2}$, for any $t_1, t_2 \in \mathbb{R}^+$. In other words, e_t^x is a function of p_t. We will use the following notation $e_t^x = e^x(p_t)$.

Proposition 3. *The patent race with private information has a unique symmetric Markov perfect equilibrium.*

The proof of the result is provided in Appendix C. Solving the ODE (2) is complicated by the fact that it is not an initial value problem: whilst we know $p_0 = \hat{p}$, we do not know the values of v_0^1 and v_0^0, and an error of the initial guess grows exponentially going forward in time. On the other hand, solving backwards in time, the error of the guess of p_t would be a problem.

[7] Symmetric in the sense that both players have the same strategies.

I show the uniqueness of the solution of the ODE (2) by proving that the system of equations has a unique critical point (Lemma 16); that every solution has to converge to this point (Lemma 18); and that there is a unique direction in which it can occur (Lemma 17).

3.2 Effort Over Time

Knowing that there is a unique equilibrium, we can discuss its properties:

Proposition 4. *In a patent race with private information, each player drops research effort over time until he makes the first breakthrough; then his effort jumps up and keeps increasing.*

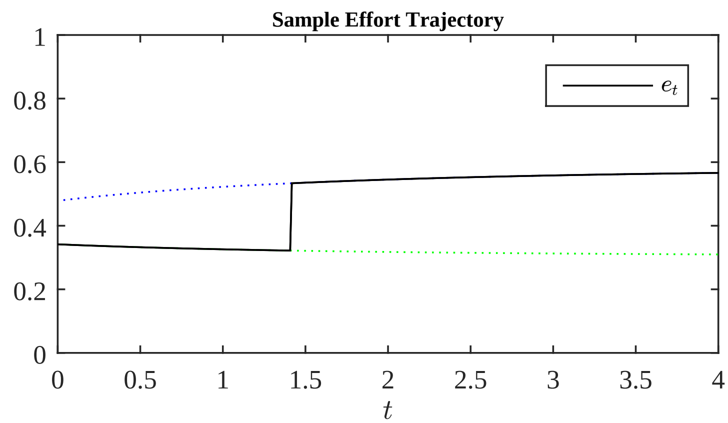

Figure 1: An example of a player's effort over time; the rise occurs as the player makes the first breakthrough.

The proposition consists of three statements about the equilibrium effort levels: (1) A successful player gets increasingly rivalrous, i.e. e_t^1 increases over time (Lemma 25). This is quite intuitive as the rival is increasingly likely to be successful over time, which means that winning the patent from state 1 is less and less likely, making the successful player exert increasing effort. (2) An unsuccessful player gives up over time, i.e. e_t^0 decreases over time. This result is a bit less straightforward; it follows from the fact that the continuation value v_t^1 of a successful player drops faster than the continuation value v_t^0 of an unsuccessful player. (3) a successful player is always more rivalrous than an unsuccessful one, i.e. $e_t^1 > e_t^0$. The results are illustrated in Figure 1.

4 One Player Known to be Successful

This section deals with situations in which one player is known to be successful. Without loss of generality, assume that the player known to be successful is player A. In other words, player A is in state 1 and it is common knowledge, while player B is expected to be in state 1 with probability $\hat{p}$ at $t = 0$. In this setting, we can analyze which of the two players is better off. The results obtained in this section have significance on their own, but their main importance is to prepare the basis for the analysis of the game in which players have the option to reveal their success.

4.1 Unique Equilibrium

Substituting $p^A = 1$ in the system of ODEs (1), we can characterize the equilibria of this game.

Proposition 5. *Every equilibrium in a patent race with private information and player A being known to be successful is characterized by the following system of ODEs*

$$\begin{aligned} -\dot{v}_t^{1A} &= \tfrac{\alpha}{2}(e_t^{1A})^2 - (r + p_t^B e_t^{1B}) v_t^{1A} \\ -\dot{v}_t^{1B} &= \tfrac{\alpha}{2}(e_t^{1B})^2 - (r + e_t^{1A}) v_t^{1B} \\ -\dot{v}_t^{0B} &= \tfrac{\alpha}{2}(e_t^{0B})^2 - (r + e_t^{1A}) v_t^{0B} \\ \dot{p}_t^B &= (1 - p_t^B)(e_t^{0B} - p_t^B e_t^{1B}), \end{aligned} \tag{3}$$

along with the identities

$$e_t^{1A} = \tfrac{1}{\alpha}(v - v_t^{1A}),\ e_t^{1B} = \tfrac{1}{\alpha}(v - v_t^{1B}),\ e_t^{0B} = \tfrac{1}{\alpha}(v_t^{1B} - v_t^{0B}),$$

the initial condition $p_0^B = \hat{p} \in [0, 1)$, and inequalities $v_t^{1A}, v_t^{1B} \in [0, v), v_t^{0B} \in [0, v_t^{1B}]$, and $p_t^B \in [0, 1]$, for all $t \geq 0$.

Following a similar line of reasoning as in Section 3, we prove the existence of a unique equilibrium in this game.

Proposition 6. *The patent race with private information and one player known to be successful has a unique symmetric Nash equilibrium.*

4.2 Effort Over Time

Knowing that there is a unique equilibrium in the game, we can discuss its properties. First, we look at how players' efforts evolve over time.

Proposition 7. *Suppose player A is known to be successful, while player B's state is unknown. Then player A increases his effort e_t^{1A} over time. As a result, player B decreases his effort over time until making the first breakthrough, after which his effort increases and continues to increase. That is, e_t^{0B} decreases over time, $e_t^{0B} < e_t^{1B}$ for all $t \geq 0$, and e_t^{1B} increases over time.*

This result is analogous to the one from the symmetric version of the game with both players in an unknown state. However, in this case the result can be found surprising, because player B changes his effort over time even though his belief about the state of his rival is fixed. The result is driven by the second-order beliefs – player B knows that he is increasingly being expected to be already in state 1, and thus player A becomes increasingly rivalrous over time.

4.3 Who is better off?

The information asymmetry gives rise to a number of questions. In particular, when both of the players have made a breakthrough, but only one of them is known for that, then we can ask which of them invests in R&D more aggressively, and who is better off.

Proposition 8. *Suppose that both players A and B are successful, but only player A is known for that. Then the informed player B invests in R&D more than player A; that is,* $e_t^{1B} > e_t^{1A}$*. As a result, the continuation value of player A is higher than that of player B,* $v_t^{1A} > v_t^{1B}$*.*

Proposition 8 shows that when both players are successful, the uninformed player is more optimistic. Next, we show that conversely the informed player is better off in that situation.

Proposition 9. *Suppose that both players A and B are successful, but only player A is known for that. Then the informed player B is better off than the uninformed player A. In other words* $v_t^{1A,(B)} < v_t^{1B}$*, where* $v_t^{1A,(B)}$ *is player A's continuation value from player B's perspective.*[8]

Proposition 9 shows that the player with more information is better off. Such a result is not as trivial as it might seem at first glance. On the one hand, having extra information gives a player the option (not obligation) to use the knowledge. On the other hand, being known about having extra knowledge can be to the player's disadvantage.

5 Patent Race with Optional Revelation

In this section, I address the main question of this paper: "Do players want to reveal their success? In other words, after a player has completed the first stage of the R&D, is it to his advantage when his rival knows about it?"

In this section, I answer these questions by providing an implicit characterization of equilibria in the patent race with revelation, and accompany it by explicit results obtained by the use of numerical methods. Any result that uses numerical methods in its proof is clearly marked as a numerical result.

The structure of incentives is such that the revelation deters an unsuccessful rival, but it accelerates the rival that is successful. As a consequence, whether a player should reveal depends on how likely he expects the rival to be unsuccessful, for how long, and whether the rival plans to reveal breakthroughs. A brief preview of the results is in the Figure 2.

This section is organized in the following way: First, I explain the game structure and the solution concept. Then, I characterize two types of equilibria in pure strategies: the *no-revelation equilibrium* and the *instant-revelation equilibrium*, After expanding the framework to be able to describe mixed strategies, I characterize the *mixed-revelation equilibrium*, in which players randomize over revelation up to a certain time T. Finally, I show that there is no other type of equilibrium with a use of numerical methods.

[8] Player B knows that both players are successful, while player A knows it only about himself.

Player A's strategy (assuming B has not revealed):

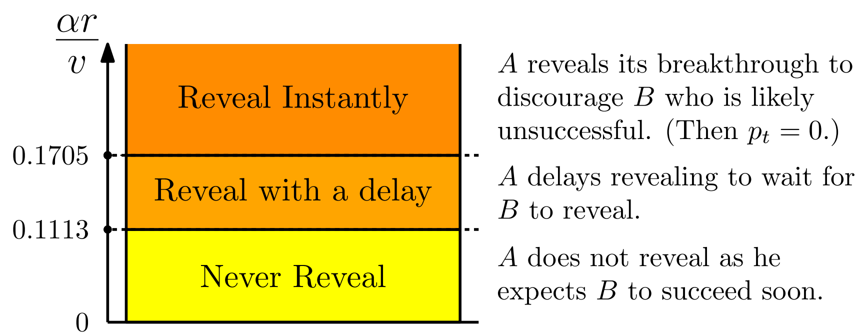

Figure 2: A preview of the results: Firms have the incentive to reveal their breakthroughs only when the research is difficult enough, as otherwise the rival of the successful firm would be likely to catch up promptly and the revealed information would increase the rivalry of the firms. Note that a delay is equivalent to using a mixed strategy over revealing breakthroughs.

5.1 Game Structure

In contrast to the private information game, in which players only silently choose their research efforts, the information revelation game allows for strategic interactions. In fact, by revealing, a player changes the incentives of his rival to reveal.

We restrict our attention to *Markov perfect Bayesian equilibria* (MPBE) in which players' belief profile (p^A, p^B) is the payoff relevant state. This simplification allows us to use the solution method of backward induction.[9]

5.1.1 Sub-games

I refer to a player being *successful* whenever he has reached state 1. Revealing then always means revealing success.

Solving the game using backward induction involves the following steps:

First, if both players have revealed already, the sub-game is equivalent to the complete information game with both players being in state 1. Hence, v^{11} (defined in the complete information version of the game) is the continuation value after both players reveal.

Second, suppose that both players are in state 1, but only j has revealed it. Then $p_t^j = 1$, $p_t^{-j} < 1$, and player $-j$ has the option to reveal his success, yielding him continuation value

[9] This assumption simplifies the analysis, but it does not necessarily rule out any of the Nash equilibria that this game might have. The analysis of the MPBE is simpler than the one of Nash equilibria, because it excludes all the strategies in which a player would respond to the rival's actions by punishing him by acting sub-optimally. In the patent race game with revelation, the only observable action of the rival is the revelation. In a Nash equilibrium, a player could potentially choose unreasonably high effort after his rival's revelation, to dissuade the rival from revealing at the first place. In contrast, in a Markov perfect equilibrium, players always need to take optimal actions regardless of the past and even in scenarios that never occur. However, it seems that a player benefits from the revelation of his rival, and thus he would have no reason to punish his rival in any Nash equilibrium anyways.

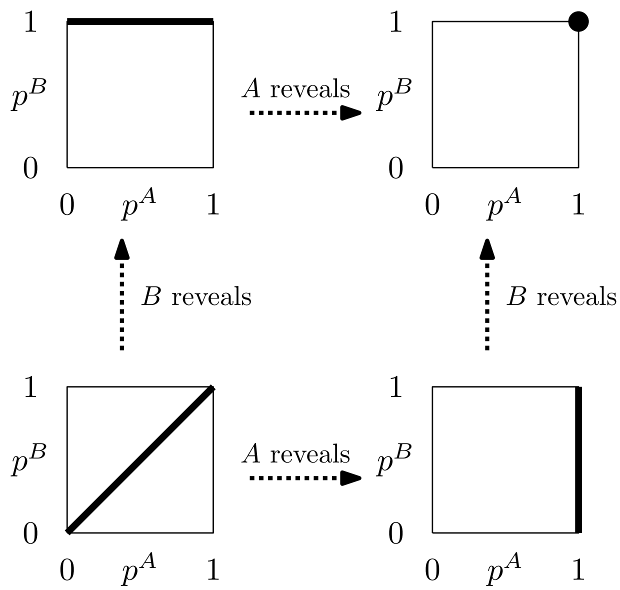

Figure 3: Four different scenarios based on who has revealed a breakthrough. These scenarios are characterized by the combination of posterior beliefs – whenever $p^j = 1$, it means that player $j \in \{A, B\}$ has revealed success. The solution method of backwards induction is used, starting with the scenario in which both players have revealed success already and ending by the analysis of the scenario before anyone has revealed.

v^{11}. I will show that player $-j$ never wants to reveal success, and thus the continuation values in this game are the same as in the private information game with player j being known to be successful.

Last, knowing what players do in the described sub-games, I study the incentives to reveal before anyone has revealed. I find three types of symmetric equilibria: for r small, players never reveal; for r large, the first player to get success reveals it immediately; and for r moderate, each of the players would reveal the arrival of his success only with a certain probability that is decreasing in time and eventually becomes zero.[10]

Throughout this section, I refer to the value function $v^{y,j}(p^A, p^B), y \in \{0, 1\}, j \in \{A, B\}$, and $p^A, p^B \in [0, 1]$ defined previously as the continuation value in the patent race without revelation of player j who is in state y, while the beliefs are given by the profile of posteriors (p^A, p^B). Note that since we only consider symmetric games, $v^{y,j}(p^A, p^B) = v^{y,-j}(p^B, p^A)$. I often use this fact and write $v^{yB}(1, p)$ in place of $v^{yA}(p, 1)$, since the case of player A being known to be successful was closely studied in Section 4.

5.1.2 Never Reveal Second

I begin by considering the situation in the patent race with revelation in which one player has already revealed success.

[10]Or he reveals with a delay, as will be discussed later.

Proposition 10. *A player never reveals success after observing the rival's revelation.*

The fact that not revealing second is equilibrium follows from the fact that $v^{1B}(1,p) > v^{1B}(1,1) = v^{11}$ for any $p \in [0,1)$.[11] Showing that not revealing second is indeed the only equilibrium strategy is more elaborate.

Proposition 10 implies that after a player $j \in \{A, B\}$ reveals success, then the other player will never reveal, and so the game continues as in the private information version of the game with player j known to be successful. As a result:

Corollary 1. *If player A reveals success before his rival does so, then he obtains the continuation value $v^{1A}(1, p_t^B)$, where p_t^B is his current belief about player B being successful. Simultaneously, player B obtains continuation value $v^{yB}(1, p_t^B)$, where y is player B's actual state. (The situation in which player B reveals is analogous.)*

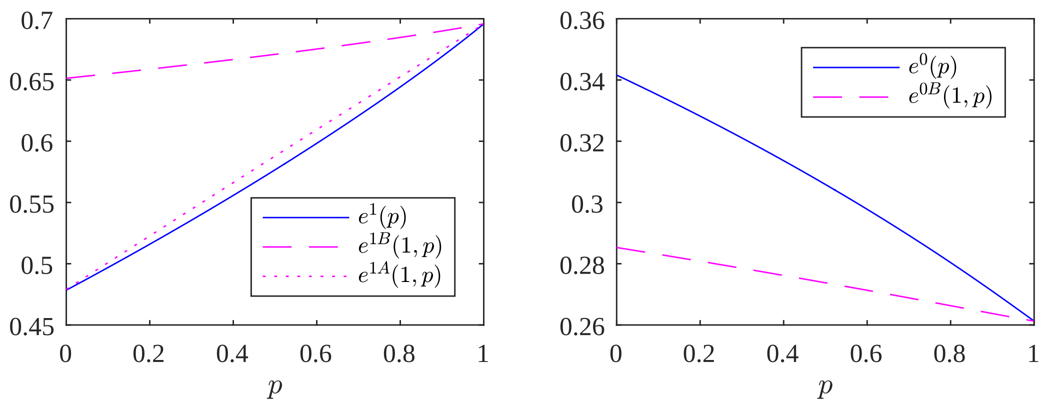

Figure 4: Illustration of the effect of player A's revelation on the rival's effort. The rival's effort if A does not reveal is the solid line, and if he does it is the dashed line. (Values for $r = 0.1, \alpha = v = 1$.)

Knowing how the game continues after either of the players reveals success allows us to discuss the incentives for revelation that a player has before any of them have revealed. It can be observed numerically that for any choice of the parameters $e^{0B}(1,p) < e^0(p)$, and $e^{1B}(1,p) > e^1(p)$, for all $p \in [0,1)$, as illustrated in Figure 4. This means that player A's revelation discourages the rival's effort so long as he is in state 0; however, once the rival makes a breakthrough, the information about player A's success only makes him choose a higher effort.

5.2 Pure-strategy Equilibria

I first focus on the two extreme strategies in which players either never reveal or they reveal breakthroughs instantly.

[11] Recall that v^{11} denotes the continuation value when both players are known to be successful.

5.2.1 No-revelation Equilibrium

We are ready to characterize the equilibrium in which none of the players reveals.

Definition 1. Define *no-revelation equilibrium* as a symmetric equilibrium in the patent race with optional revelation in which both players have the strategy to never reveal.

If players follow the no-revelation equilibrium strategy, none of them reveals and thus the game evolves as described in Section 3. Should either of the players deviate and reveal, the continuation values will be as described in Corollary 1.

Proposition 11. *There is a no-revelation equilibrium if and only if $v^{1A}(1,p) \leq v^1(p)$, for all $p \in [0, p_*]$, where p_* is the steady-state value of p in the private information version of the game. In that case, such equilibrium is equivalent to the equilibrium of the patent race with private information.*

Lemma 2 (partially numerical)**.** *The inequality $v^{1A}(1,p) \leq v^1(p)$ holds for all $p \in [0,1)$ if and only if $v^{1A}(1,0) \leq v^1(0)$. If $v^{1A}(1,0) > v^1(0)$, then the following single crossing condition holds: there exists $\bar{p} \in (0,1)$ such that $v^{1A}(1,\bar{p}) = v^1(\bar{p})$; $v^{1A}(1,p) > v^1(p)$ for $p < \bar{p}$; and $v^{1A}(1,p) < v^1(p)$ for $p > \bar{p}$.*

The validity of this lemma can be tested numerically for any $r \geq 0$. Intuitively, the lemma holds for the following reason: The functions $p \mapsto v^{1A}(1,p)$ and $p \mapsto v^1(p)$ intersect at $p = 1$ by definition (they are both equal to v^{11} there). Of the two functions, the function $p \mapsto v^1(p)$ has higher curvature as it corresponds to the value function of the posterior of both players changing simultaneously, while in the case of the function $p \mapsto v^{1A}(p)$, only the posterior about player B is changing with p.

The Proposition 11 characterizes the conditions for the existence of a no-revelation equilibrium by referring to value functions for which we do not have explicit formulas. Due to Lemma 2, we only need to verify the inequality $v^{1A}(1,0) \leq v^1(0)$. It can be shown numerically that this inequality is satisfied whenever $r' = \frac{\alpha r}{v}$ is smaller than approximately 0.1113.

5.2.2 Instant-revelation Equilibrium

On the other side of the spectrum is the equilibrium in which players reveal instantly.

Definition 2. Define an *instant-revelation equilibrium* as a symmetric equilibrium in the patent race with optional revelation in which both players have the strategy to reveal instantly unless the rival has already revealed. Once either of the players reveals, the game continues as in the private information version of the game with one player known to be successful.

Interestingly, in an instant-revelation equilibrium the game is static until either of the players reveals. This is because until either of the players reveals, they are certain to be both at the starting point. An instant-revelation equilibrium exists whenever a player will not be tempted to deviate by omitting revelation in it. That condition gives us the following implicit characterization of the existence of an instant-revelation equilibrium.

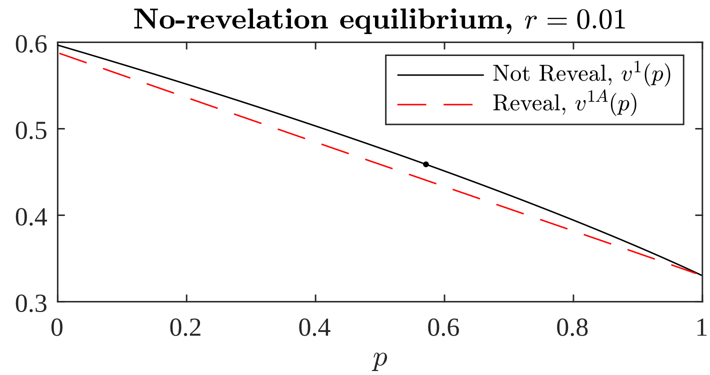

Figure 5: The continuation value of a successful player in a no-revelation equilibrium in comparison with the continuation value that a player obtains after revealing success. Both are functions of the posterior belief p_t, which grows from zero up to its steady-state value p_* (dot in the graph). Comparison of the two continuation values matters only left from the dot. Evaluated for $r' = 0.01$.

Proposition 12. *There is a unique instant-revelation equilibrium if and only if*

$$0 \geq \frac{\alpha}{2}\left(v - v^{1A}(1,0)\right)^2 + e^0_\bullet v^{1B}(1,0) - (r + e^0_\bullet)v^{1A}(1,0), \tag{4}$$

where $e^0_\bullet \in (0, v^{1A}(1,0))$ *is the unique positive solution of the quadratic equation*

$$0 = \frac{\alpha}{2}\left(e^0_\bullet\right)^2 + e^0_\bullet v^{0B}(1,0) - (r + e^0_\bullet)(v^{1A}(1,0) - e^0_\bullet). \tag{5}$$

In that case, each player exerts constant effort $e^0_\bullet$ *until either of them makes a breakthrough.*

It can be shown numerically that the inequality (4) holds if and only if r' is above the threshold of approximately 0.1707.

Notice that, according to the numerical results, the no-revelation and the instant-revelation equilibria cannot exist simultaneously (for a given value of r'). The intuition for the result is that revealing is less attractive when the rival has the strategy to reveal. When r' is above 0.1113, a player would have an incentive to deviate by revealing in the no-revelation equilibrium, and yet he would not have an incentive to reveal in the instant-revelation equilibrium so long as r' is below 0.1707. Next, I show that a mixed-revelation equilibrium is present for r' in between the two thresholds. The situation is depicted in Figure 6.

5.3 Mixed-strategy Equlibria

5.3.1 General Characterization of Equilibria in Mixed Strategies

So far, I have discussed two extreme types of equilibria in pure strategies. To consider equilibria in mixed strategies, it is necessary to clarify what a player's strategy is in this game.

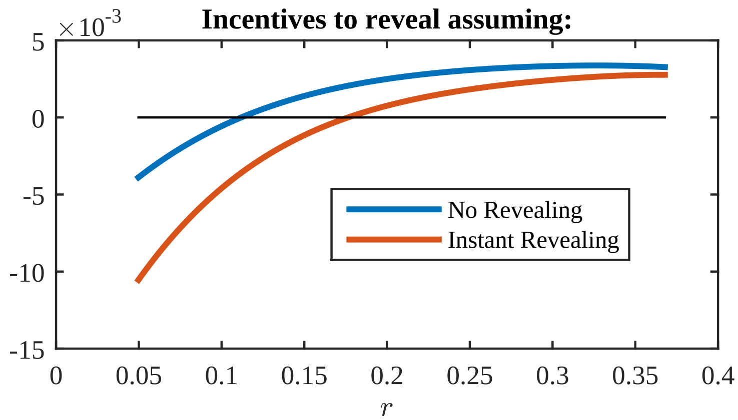

Figure 6: The incentives to reveal as a function of the research difficulty $r' = \frac{\alpha r}{v}$ and the assumption about players' strategies. When both curves are below zero, then no-revelation equilibrium is present. When both curves are above zero, then an instant-revelation equilibrium is present. In between, a mixed-revelation equilibrium is present, as will be shown later.

Definition 3. Player j's strategy before anyone has revealed is given by the profile of right-continuous functions $\{(e^{1j}_{\bullet t}, e^{0j}_{\bullet t}, \theta^j_{\bullet t}), t \geq 0\}$, where $(e^{1j}_{\bullet t} \in (0, +\infty)$ and $e^{0j}_{\bullet t} \in (0, +\infty)$ are player j's effort $e^{yj}_{\bullet t}$ in state 1 and 0, respectively, and $\theta^j_{\bullet t} \in [0, +\infty]$ is the hazard rate with which player j is expected to reveal by his rival $-j$.[12] The strategy has to satisfy the Markov property, which in the case of a symmetric equilibrium, means that there exist functions $e^1_\bullet(\cdot)$, $e^0_\bullet(\cdot)$ and $\theta_\bullet(\cdot)$, such that

$$e^{1j}_{\bullet t} = e^1_\bullet(p_t), \;\; e^{0j}_{\bullet t} = e^0_\bullet(p_t), \;\; \theta^j_{\bullet t} = \theta_\bullet(p_t), \quad \forall t \geq 0.$$

For regularity, I require the functions $e^{1j}_\bullet$, $e^0_\bullet$ and $\theta_\bullet$ to be piece-wise continuous. I denote the associated continuation values as $v^{0j}_{\bullet t}$ and $v^{1j}_{\bullet t}$.

Note that the definition of player j's strategy specifies with what hazard rate player j reveals from the perspective of his rival, but it does not specify whether he reveals a breakthrough that he has just made or one that he made before. One potential way to interpret player j's strategy is that he reveals a breakthrough upon its arrival with the probability $\Theta^j_{\bullet t} \in [0, 1]$. Then, $\theta^j_{\bullet t} = (1 - p^j_t) e^{0j}_{\bullet t} \Theta^j_{\bullet t}$, because from $-j$'s perspective player j is with probability $1 - p^j_t$ in the state 0, in which case he makes a breakthrough with the hazard rate $e^{0j}_{\bullet t}$, and he consequently reveals it with probability $\Theta^j_{\bullet t}$. In what follows I clarify that every equilibrium strategy can be interpreted this way (we need to ensure that $\Theta^j_{\bullet t} \leq 1$).

We first show that the case of $\theta^j_{\bullet t} = +\infty$ does not need to be considered

Lemma 3. *In any symmetric equilibrium, the hazard rate $\theta^j_{\bullet t}$ is finite.*

Using Bayes' law we obtain the following law of motion for the posterior beliefs when players have the option to reveal:

[12] The rival has no information about player j's state, and thus the hazard rate is not conditioned on it.

Lemma 4. *The law of motion for p_t^j before either of the players reveals is $\dot{p}_t^j = (1-p_t^j)(e_{\bullet t}^{0j} - p_t^j e_{\bullet t}^{1j} - \theta_{\bullet t}^j)$.*

Finally, we take advantage of the Markov Property to show that p_t has to be nondecreasing:

Lemma 5. *In any symmetric equilibrium, $\dot{p}_t \geq 0$ and $\theta_{\bullet t} \leq e_{\bullet t}^0 - p_t e_{\bullet t}^1$, for any $t \geq 0$.*

To be able to discuss a player's strategy, I need to specify exactly what I mean by a mixed-strategy in this context:

Definition 4. We say that at time t player j:

- *does not reveal* if $\theta_{\bullet t}^j = 0$;
- *randomizes over revelation* if $\theta_{\bullet t}^j \in (0, e_{\bullet t}^{0j})$;
- *reveals with certainty* if $\theta_{\bullet t}^j = e_{\bullet t}^{0j}$.[13]

Lemma 6. *In a symmetric equilibrium other than the instant-revelation equilibrium, players never reveal with certainty.*

The general equilibrium conditions are as follows:

Lemma 7. *In any symmetric equilibrium in the patent race with optional revelation other than the instant-revelation equilibrium, the strategy of either player before anyone has revealed is characterized by the following system of ODEs:*

$$-\dot{v}_{\bullet t}^1 = \frac{\alpha}{2}(e_{\bullet t}^1)^2 + \theta_{\bullet t} v^{1B}(1, p_t) - (r + \theta_{\bullet t} + p_t e_{\bullet t}^1) v_{\bullet t}^1 \quad (6)$$

$$-\dot{v}_{\bullet t}^0 = \frac{\alpha}{2}(e_{\bullet t}^0)^2 + \theta_{\bullet t} v^{0B}(1, p_t) - (r + \theta_{\bullet t} + p_t e_{\bullet t}^1) v_{\bullet t}^0 \quad (7)$$

$$\dot{p}_t = (1 - p_t)(e_{\bullet t}^0 - p_t e_{\bullet t}^1 - \theta_{\bullet t}), \quad (8)$$

where $e_{\bullet t}^1 = \frac{1}{\alpha}(v - v_{\bullet t}^1)$, $e_{\bullet t}^0 = \frac{1}{\alpha}(v_{\bullet t}^1 - v_{\bullet t}^0)$, and the initial condition $p_0 = 0$, and the inequalities $0 \leq v_{\bullet t}^0 \leq v_{\bullet t}^1 < v$ and $p_t \in [0, 1]$, $\theta_{\bullet t} \in [0, e_{\bullet t}^0 - p_t e_{\bullet t}^1]$, for all $t \geq 0$. In addition to that, $v_{\bullet t}^1 \geq v^{1A}(1, p_t)$, and $\theta_{\bullet t} = 0$ whenever the inequality is slack.

By comparing the player's continuation value of revealing at time t and postponing it by $\Delta t > 0$, we obtain the following necessary condition for a player to be randomizing over revelation.

Corollary 2. *In a symmetric equilibrium other than the instant-revelation equilibrium, $\theta_{\bullet t} > 0$ implies:*[14]

$$-v_{\partial p}^{1A}(1, p_t)\dot{p}_t = \frac{\alpha}{2}(e^{1A}(1, p_t))^2 + \theta_{\bullet t} v^{1B}(1, p_t) - (r + \theta_{\bullet t} + p_t e^{1A}(1, p_t)) v^{1A}(1, p_t). \quad (9)$$

Note that the necessary condition does not have to be sufficient as the player might not want to reveal at all.

[13] which, by Lemma 5, can occur only when $p_t^j = 0$.

[14] $v_{\partial p}^{1A}(1, p_t)$ denotes the derivative of the function $p \mapsto v^{1A}(1, p)$

5.3.2 Mixed-revelation Equilibrium

The expectation of the rival's revelation discourages the player's own revelation. As a result, there is a range of values of research difficulty $r' = \frac{\alpha r}{v}$ for which neither of the pure equilibria exists: In a no-revelation equilibrium, a player would be tempted to reveal, yet he would not have sufficient incentives to reveal in an instant-revelation equilibrium. Then, an equilibrium can be found only in mixed strategies, so that each player reveals exactly with the probability that keeps his rival indifferent to revealing.

Let us focus attention on a special type of symmetric equilibrium with mixed-strategy revealing in which players mix over revelation (given that none of them has revealed yet) until a certain deadline, after which they do not reveal at all:

Definition 5. Define a *mixed-revelation equilibrium* as a symmetric equilibrium in the patent race with optional revelation in which there is $T \in (0, +\infty)$, such that $\theta_{\bullet t} > 0$ for all $t \in [0, T)$ and $\theta_{\bullet t} = 0$ for all $t \in [T, +\infty)$.

Players stop revealing at the point when revealing yields the same continuation value as not revealing, assuming none of the players will reveal later on:

Lemma 8. *If $T > 0$ is the time at which players stop revealing definitively in a symmetric equilibrium, then $v^{1A}(1, p_T) = v^1(p_T)$.*

Proposition 13. *Mixed-revelation equilibrium is uniquely characterized as follows: There is $T > 0$ such that $v^{1A}(1, p_T) = v^1(p_T)$, and for all $t \in [0, T)$,*

$$\theta_{\bullet t} = \frac{\frac{\alpha}{2}(e^{1A}(1,p_t))^2 - (r + e^0_{\bullet t})v^{1A}(1,p_t) + (e^0_{\bullet t} - p_t e^{1A}(1,p_t))\tilde{v}^{1A}_t}{\tilde{v}^{1A}_t - v^{1B}(1,p_t)} \quad (10)$$

$$-\dot{v}^0_{\bullet t} = \frac{\alpha}{2}(e^0_{\bullet t})^2 + \theta_{\bullet t} v^{0B}(1,p_t) - (r + \theta_{\bullet t} + p_t e^{1A}(1,p_t))v^0_{\bullet t} \quad (11)$$

$$\dot{p}_t = (1 - p_t)(e^0_{\bullet t} - \theta_{\bullet t} - p_t e^{1A}(1,p_t)), \quad (12)$$

in which $\tilde{v}^{1A}_t = v^{1A}(1,p_t) + (1-p_t)v^{1A}_{\partial p}(1,p_t)$, and $e^0_{\bullet t} = \frac{1}{\alpha}(v^{1A}(1,p_t) - v^0_{\bullet t})$. For all $t \geq T$, the game continues as the game without the option to reveal; that is, $\theta_{\bullet t} = 0$, $v^1_{\bullet t} = v^1(p_t)$, $v^0_{\bullet t} = v^0(p_t)$, and the law of motion of p_t is given by equation (2).

The above characterization of the mixed-revelation equilibrium also provides the prescription for calculating it: Start at $\overline{p} \in (0,1)$ such that $v^{1A}(1,\overline{p}) = v^1(\overline{p})$, and put $v^0_\bullet(\overline{p}) = v^0(\overline{p})$. Then solve the ODE $v^0_{\bullet\,\partial p}(p) = \frac{\dot{v}^0_\bullet(p)}{\dot{p}(p)}$ from $\overline{p}$ down to 0; having solved for $v^0_\bullet(\cdot)$, we can put $p_0 = 0$ and find p_t using the law of motion (12) until p_t reaches the value $\overline{p}$, and define this time as T.

It can be shown numerically that for any value of the parameter r', both the numerator and the denominator of (10) are positive, and thus $\theta_{\bullet t}$ is always well defined. Potential trouble would be if $\dot{p}(p)$ would reach zero at some point. It can be shown numerically that this is the case if and only if the instant-revelation equilibrium exists. Define $(\underline{p}, \underline{v}^0_\bullet) \in [0,1) \times [0, v^{1A}(\underline{p}))$ to be the steady-state of the system of ODEs (11)-(12). It can be shown numerically that such a steady-state exists (and is unique) if and only if the instant-revelation equilibrium

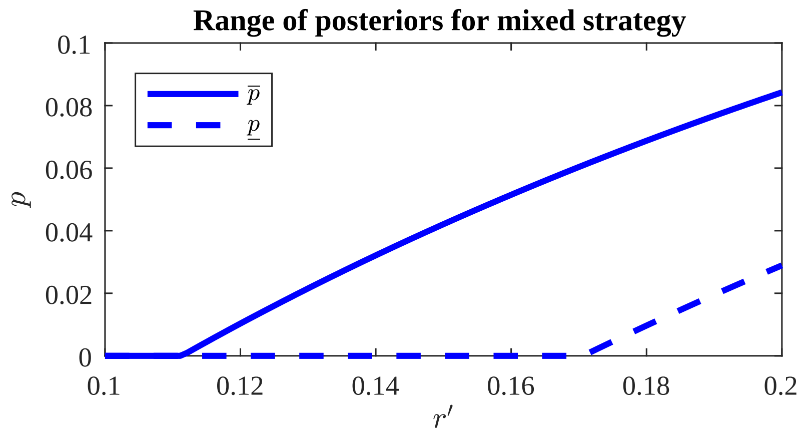

Figure 7: The range $[\underline{p}, \overline{p}]$ of p at which players might be randomizing over revelation, as a function of the research difficulty $r' = \frac{\alpha r}{v}$. For $r' < 0.1113$ players never mix over revelation because they are always better off by not revealing. However, for $r' > 0.1707$ players would randomize only if $p_0 \in (\underline{p}, \overline{p})$, which is not consistent with the assumption that both players start in state 0 with certainty. Only for r' in between the two bounds does a mixed-revelation equilibrium exit.

exists. How $\underline{p}$ varies with the research difficulty is shown in Figure 7. Consider the case that $\underline{p} > 0$. Since we assume $p_0 = 0$, players can only reveal with certainty and p_t remains at zero. However, the situation would be different if we considered that players initially had made a breakthrough with a positive probability. If $p_0 \in (\underline{p}, \overline{p})$, then players randomize over revelation until they stop revealing at all when p_t reaches $\overline{p}$. On the other hand, if $p_0 \in (0, \underline{p})$, then players randomize over revelation until p_t reaches 0, after which they have the strategy to reveal instantly.

The mixed-revelation equilibrium cannot coexist with the previously discussed equilibria in pure strategies:

Lemma 9 (partially numerical)**.** *A mixed-revelation equilibrium exists if and only if there is no pure strategy equilibrium (the no-revelation equilibrium or the instant-revelation equilibrium).*

5.3.3 Three Types of Equilibria

It can be shown that the only equilibrium involving randomizing over revelation is the mixed-revelation equilibrium.

Lemma 10 (partially numerical)**.** *If players ever randomize over revelation, then they do so on a time interval $t \in [0, T)$ for some $T > 0$.*

Putting everything together, we obtain the final result that was illustrated in Figure 2.

Proposition 14 (partially numerical)**.** *The patent race with optional revelation has a unique equilibrium. Depending on the research difficulty $r' = \frac{\alpha r}{v}$, the type of the equilibrium is:*

- no-revelation equilibrium *for* $r' \in [0, r^N]$*;*
- mixed-revelation equilibrium *for* $r' \in (r^N, R^I)$*;*
- instant-revelation equilibrium *for* $r' \in [r^I, +\infty)$*,*

where the thresholds are approximately $r^N \approx 0.1113$ *and* $r^I \approx 0.1707$*.*

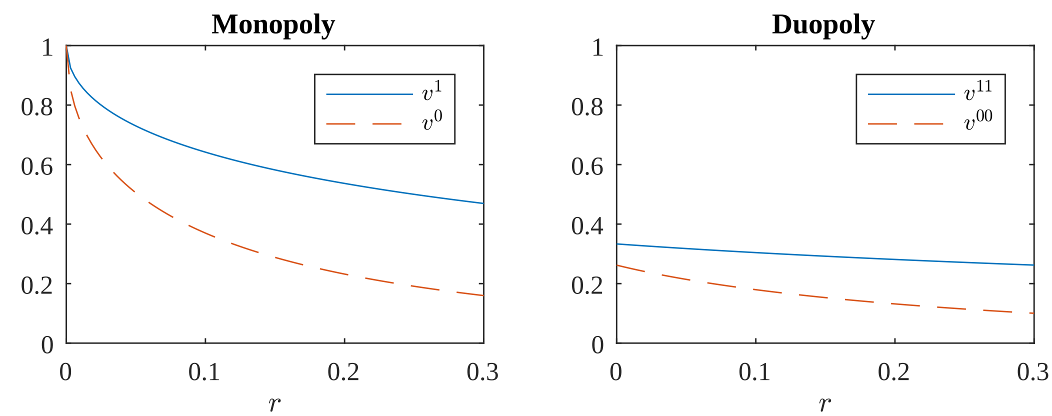

Figure 8: These graphs show how a player's (expected) payoff varies with the research difficulty r' ($v = 1$). Value v^1 and v^0 is the payoff of a monopolist one step and two steps away from patenting, respectively; v^{11} is the payoff of each of the players when they are both one step away from patenting; and finally v^{00} is the payoff of each of the players when both of them are two steps away from patenting (assuming their state is common knowledge).

Although the research difficulty parameter r' cannot be measured, it can be estimated from players' expected payoffs as shown in Figure 8. For example, the results of Proposition 14 could be summarized as: players reveal instantly if the research difficulty is such that the expected payoff of a monopolist two steps away from patenting would be less than 26.3% of the value of the patent; whilst the players never reveal if that payoff was above 34.8% of the value of the patent.

6 Welfare Comparison, Extensions and Robustness

6.1 Welfare Comparison

We have seen how privacy about a firm's progress towards a patent changes the dynamics of the patent race. The question then is how does it impact the welfare of the firms. On the one hand, having less information about their rivals means that firms make less optimal choices of their research efforts. On the other hand, the lack of information might decrease the degree of rivalry of the firms.

Are the firms better off under the private information setting or under the complete information setting?[15] Does the option to reveal progress increase the welfare? The answer to these questions can be found in Figure 9, which compares welfare among various information settings, depending on the research difficulty parameter $r' = \frac{\alpha r}{v}$.

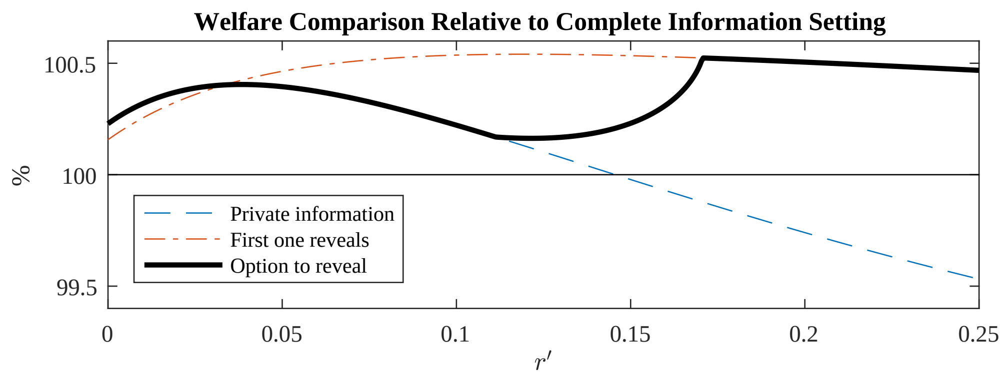

Figure 9: Comparison of welfare across various information settings. Welfare is measured in terms of a player's ex-ante expected payoff, and it is expressed as the percentage of the welfare in the complete information setting.

We can observe the following results: The private information setting is better for the firms than the complete information setting so long as the research is not too difficult (approximately $r' < 0.1456$). Interestingly, the information setting in which only one player reveals progress (instantly) is always better for the players than the complete information setting, and it is better even than the private information setting unless the research difficulty is very low ($r' < 0.0334$).

Finally, we can observe that the patent race in which firms voluntarily reveal their progress is always better for the firms than the one in which revelation was mandatory (complete information setting) and is at last as good as the one with no option to reveal (private information setting).

An optimal policy would then be ideal to enable firms to voluntarily reveal their progress and incentivize revelation in the case of moderate research difficulty ($0.0334 < r' < 0.1707$).

6.2 Extensions and Robustness

Some of the results about players' incentives to reveal their progress, despite being quite intuitive, are unfortunately only numerical. Then it is important to determine whether the results are due to the specific choice of the model, or if they are robust.

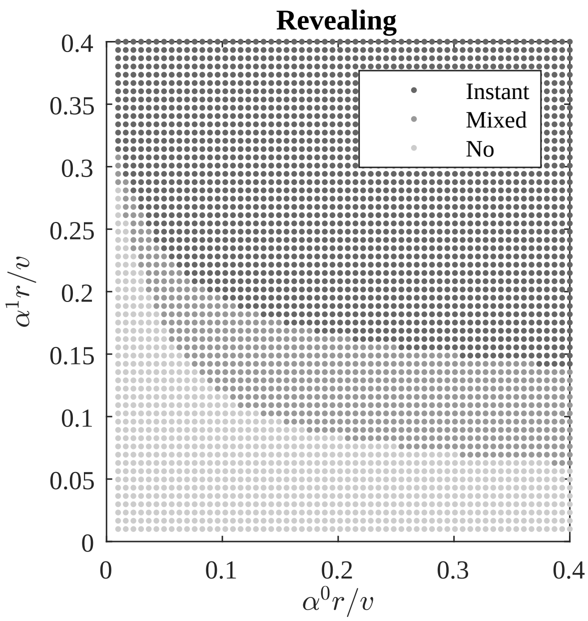

Figure 10: Classification of equilibria depending on the research difficulty of each stage of the research. The case of equal research difficulty studied in the rest of this paper corresponds to the 45 degree line.

6.2.1 Different Research Difficulty of the Stages

6.2.2 Starting Point

Throughout the paper, it has been assumed that both players initially start with having no progress (they are in state 0). As a result, their posterior beliefs about each other's progress would evolve symmetrically, unless one of the players reveals success. However, it is worth considering that one firm was known for having the initial advantage of being in state 1 with a positive probability at $t = 0$. Allowing it breaks the symmetry of the game and leads to several interesting new phenomena. Two types of observations can be made. First, the dynamics of the patent race with private information (and no option to reveal) changes, and it no longer has to be the case that a player's effort would evolve monotonically over time in each state. Second, when players have the option to reveal, the player with the initial advantage is more willing to reveal.

The main observation in Section 3 was that both players continue to decrease effort while in state 0, but they continue to increase effort if in state 1. The result was driven by the fact that the posterior beliefs $(p_t^A, p_t^B) = (p_t, p_t)$ were increasing over time. However, when one player, say A, starts "ahead" of the other, i.e. $p_0^A > p_0^B$, then p_t^A eventually becomes decreasing as it approaches its steady-state value. As a result, player B eventually becomes increasingly optimistic if in state 0 (so that e_t^{0B} becomes increasing) and decreasingly rivalrous if in state 1 (so that e_t^{1B} becomes decreasing). These results follow from the analysis

[15]Complete information in the sense that players' states are observed, while their efforts are still private.

of the Jacobian of the ODE (1) at the steady-state.

6.2.3 Asymmetric Equilibria

We have shown that for a moderate research difficulty there are no pure equilibria in the revelation game: If both players had the strategy to never reveal, then they would be tempted to reveal; and conversely, if both players had the strategy to reveal instantly, then they would be tempted not to reveal. As a result, the only symmetric equilibrium is the one in which players are randomizing over revelation to keep each other indifferent to revealing. One might expect then, that besides the mixed-revelation equilibrium, there should also be one in which one player has the strategy to reveal instantly (say player A) and the other to never reveal (player B). However, this is not possible in the symmetric game (with both players facing the same research costs and starting in state 1 with the same probability): Let us analyze the game before any of the players has revealed. Since player A is known to reveal instantly, he must be in state 0 with certainty. However, as player B does not reveal, he is increasingly likely to be already successful (in the interim state 1). As a result, player A has less incentive to reveal than player B does, as he is more likely to face a rival that is already successful. This is a contradiction with player A revealing instantly and player B never revealing being an equilibrium.

The situation becomes much more complex once we consider asymmetric versions of the game, i.e. a patent race in which the players have different effort cost and starting points (p^A and p^B). I analyze the equilibria for all the combinations of players' effort costs, $(\alpha^A, \alpha^B) \in \mathbb{R}^2_+$ and the starting posterior $p^A = p^B = 0$. [16] I find at most one equilibrium for any combination of parameters. (For some specific combinations of parameters I do not find any equilibrium, perhaps because I have not considered those types of equilibria.) Figure 11 shows regions with various types of equilibria. These types, indexed by the combination of player A and B's tendency to reveal (from 1 – never reveal, to 7 – reveal instantly), are

11 Both of the players have the strategy to never reveal, e.g. for $(\alpha^A, \alpha^B) = (0.1, 0.1)$.

16 Player A has the strategy to never reveal, whilst palyer B has the strategy to reveal instantly until a certain deadline T, after which he never reveals, e.g. for $(\alpha^A, \alpha^B) = (0.2, 0.1)$.

33 Both players are randomizing over revelation until a certain deadline T, after which the players never reveal, e.g. for $(\alpha^A, \alpha^B) = (0.15, 0.15)$.

35 Until certain time $T_1 > 0$ player A has the strategy not to reveal, whilst player B has the strategy to reveal instantly; then both of the players are randomizing over revelation until a later time $T_2 > T_1$; after which none of the players ever reveals, e.g. for $(\alpha^A, \alpha^B) = (0.2, 0.17)$.

[16] Assuming that both players value a patent equally by $v = 1$ and their discount rate is $r = 1$. Otherwise we can consider $\frac{\alpha^j r}{v^j}$ instead of α^j, $j \in \{A, B\}$. I also assume that the research difficulty is the same in both stages, i.e. that $\alpha^{j0} = \alpha^{j1}$, for $j \in \{A, B\}$.

77 Both players have the strategy to reveal instantly, e.g. for $(\alpha^A, \alpha^B) = (0.2, 0.2)$.

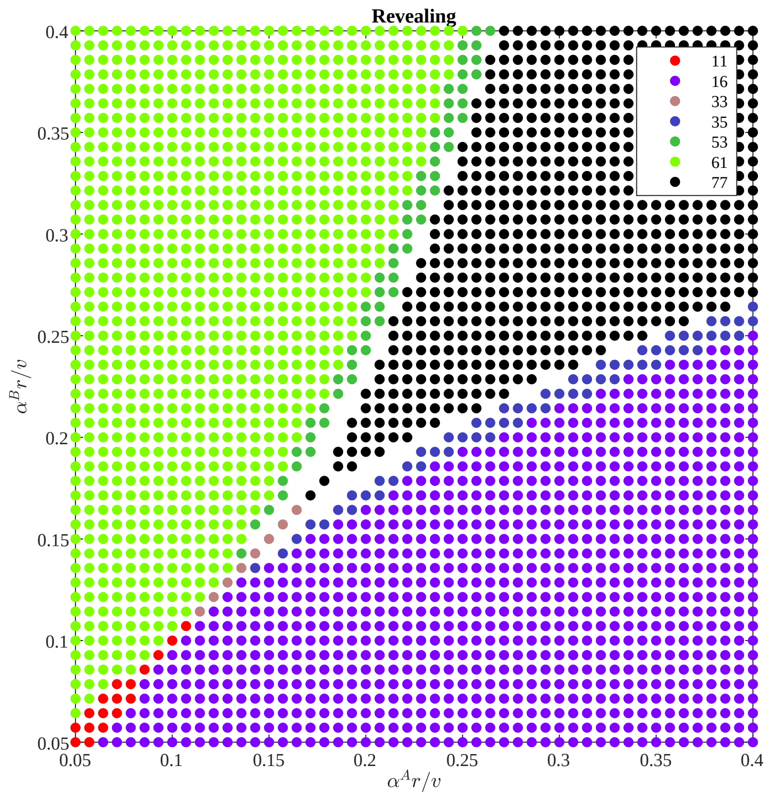

Figure 11: Classification of equilibria for varying values of parameters α^A and α^B.

6.2.4 Other Effort Cost Functions

Throughout this paper, I have assumed the quadratic effort cost function $c(e) = \alpha e^2$, as it simplifies the algebra considerably. However, the equations characterizing equilibria can be generalized to any strictly convex twice differentiable function $c(\cdot)$. Figure 12 investigates the case of the function form $c(e) = \alpha e^\gamma / \gamma$.

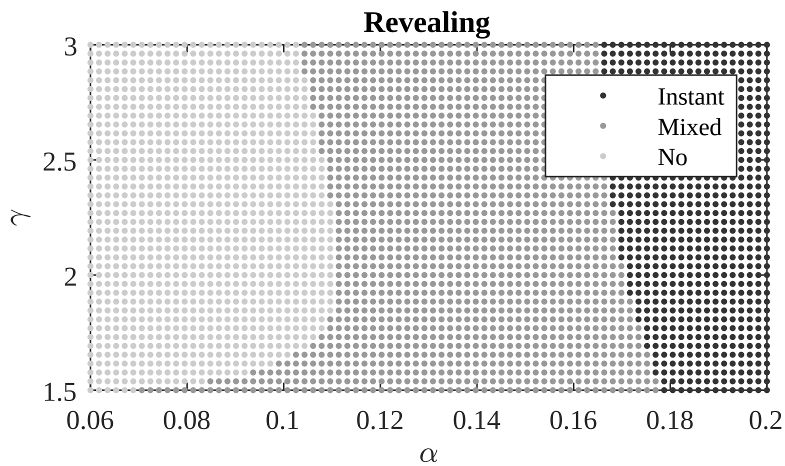

Figure 12: Classification of equilibria depending on the parameters of the effort cost function. The function form $c(e) = \alpha e^{\gamma}/\gamma$ is considered, and α and γ is varied, while $r = v = 1$. The assumption throughout the rest of this paper is that $\gamma = 2$.

7 Conclusion

This paper explores the role of private information about firms' progress in a patent race. The two primary objectives are 1) to understand how the race evolves when firms do not observe each others progress towards making a patent; 2) to investigate when firms have the incentive to reveal their success. The main takeaways are that a rival's success discourages the effort of an unsuccessful firm, but it encourages effort of a successful firm. Accordingly, a firm wants to reveal its success only if it expects its rival to be and to remain behind.

I implicitly characterize three types of equilibria in the patent race with the option to reveal breakthroughs. However, to determine which of the equilibria is present for a given choice of parameters, it is necessary to solve the given system of equations numerically. I show that the results extend even to more general settings, yet the necessity to use numerical methods is a shortcoming of the complexity of the model used. There are various ways to simplify the model, one of which would be to rewrite the model into discrete time and consider a small number of time periods. This certainly simplifies the numerical solution as the system of ODE's simplifies into a system of multivariate cubic equations. However, even if there were only three periods (the absolute minimum for the revelation of the breakthrough to matter), we would not be able to obtain a closed form solution.[17] Note that such a system of equations cannot be solved recursively, because whilst the continuation values are given by the boundary conditions in the last period, the posterior belief is given only in the first period. An alternative simplification is to allow only for low and high effort, as Gordon (2011) does. However, in that case the game might have multiplicity or no equilibria, and thus it is difficult to draw any conclusions about player's behavior. In fact, such a simplification completely changes the structure of the incentives, as then information influences a player's decision only when he is almost indifferent between low and high effort. For example, if an unsuccessful player learns that his rival is ahead, he decreases his effort in the setting with continuous effort choice, but the information would perhaps not matter to him in the setting with a binary effort choice, as he would exert low effort when being unsuccessful anyway.

In practice, firms have to consider a number of factors when deciding whether to reveal breakthroughs. The revelation might help to raise further investments, but it might also lead to technological leakage, or it might show rivals that a solution to a certain technological problem exists. I abstract from these factors and focus on the single aspect of the patent race – private information about a firm's progress and its revelation with the aim to discourage a rival.

[17]Alternatively, we would obtain a system of multivariate quadratic equations if we considered a linear effort cost function, but that still does not allow a closed form solution.

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Appendices

A The Dynamics

A.1 Law of motion for the posterior belief

Proof of Lemma 1. The posterior belief follows the Bayes Law: Take the conditioned probability p_t^j as given and assume that the game has not ended by time t. Then with probability $(1-p_t^j)$ the state is $x_t^j=0$, and with hazard rate e_t^{0j} it proceeds to the state $x_{t+\Delta t}^j=1$; with probability p_t^j the state is $x_t^j=1$ and with the hazard rate e_t^{1j} it proceeds and the game ends. Then

$$p_{t+\Delta t}^j = P[x_{t+\Delta t}^j=1|x_{t+\Delta t}^j<2] = \frac{P\left[x_{t+\Delta t}^j=1\middle|x_t^j<2\right]}{P\left[x_{t+\Delta t}^j<2\middle|x_t^j<2\right]} = \frac{(1-p_t^j)e_t^{0j}\Delta t + p_t^j(1-e_t^{1j}\Delta t)}{1-p_t^j e_t^{1j}\Delta t},$$

and so $\dot{p}_t^j = \frac{\partial}{\partial \Delta t} p_{t+\Delta t}^j\big|_{\Delta t=0} = (1-p_t^j)(e_t^{0j}-p_t^j e_t^{1j})$. □

A.2 Normalization

Proof of Proposition 2. Multiplying each of the equations in the system of ODEs 1 by α/v and dividing the first two by v we obtain the following system of ODEs:

$$\begin{aligned}
-\alpha\frac{\dot{v}_t^{1j}}{v^2} &= \tfrac{1}{2}\left(\frac{\alpha e_t^{1j}}{v}\right)^2 - \left(\frac{\alpha r}{v} + p_t^{-j}\,\frac{\alpha e_t^{1,-j}}{v}\right)\frac{v_t^{1j}}{v}\\
-\alpha\frac{\dot{v}_t^{0j}}{v^2} &= \tfrac{1}{2}\left(\frac{\alpha e_t^{0j}}{v}\right)^2 - \left(\frac{\alpha r}{v} + p_t^{-j}\,\frac{\alpha e_t^{1,-j}}{v}\right)\tfrac{v_t^{0j}}{v}\\
\frac{\alpha \dot{p}_t^j}{v} &= (1-p_t^j)\left(\frac{\alpha e_t^{0j}}{v} - p_t^j\frac{\alpha e_t^{1j}}{v}\right).
\end{aligned}$$

This system of equations is identical to the system of ODEs (1) with parameters $\hat{v}=1,\ \hat{\alpha}=1,$ and $\hat{r}=\frac{\alpha r}{v}$, and variables

$$\hat{v}_t^{xj} = \frac{v_{t\alpha/v}^{xj}}{v},\quad \hat{e}_t^{xj} = \frac{\alpha e_{t\alpha/v}^{xj}}{v},\ \text{and}\ \hat{p}_t^j = p_{t\alpha/v}^j,\quad \text{for all } x\in\{0,1\}, j\in\{A,B\}.$$

□

A.3 An Auxiliary Result

First of all, we prove some elementary properties of a function that we will use in numerous proofs.

Lemma 11. *For any given $r > 0$, the function*

$$\phi(x) := \frac{x^2}{2(1-x)} - r$$

is strictly increasing and strictly convex on $[0, 1)$ and it has a well defined inverse function ϕ^{-1} on $[0, 1]$ which is strictly increasing and concave. In addition, ϕ has a unique positive fixed point x_*. The fixed point is in the interval $x_* \in (\frac{2}{3}, 1)$, and $\phi(x) < x$ if and only if $x < x^*$ for any $x \in [0, 1]$.

Proof. Since both of the functions $x \mapsto x^2$ and $x \mapsto \frac{1}{1-x}$ are strictly increasing and strictly convex on the interval $[0, 1)$, so is their product, and consequently also the function ϕ. Further, $\phi(0) = -r < 0$ and $\phi(1_-) = +\infty$.[18]

It follows that the inverse function ϕ^{-1} is well defined on $[0, 1]$, and it is strictly increasing and strictly concave. Since $\phi(0) = -r$, $\phi'(0) = 0$, $f(1_-) = +\infty$, and ϕ is strictly convex, the function f has to intersect the identity function at a unique point x^*, and $\phi(x) < x$ if and only if $x < x^*$. Since $\phi(\frac{2}{3}) = \frac{2}{3} - r < \frac{2}{3}$, necessarily $x^* > \frac{2}{3}$. □

B Complete Information Model

Without loss of generality, assume $v = 1$ and $\alpha = 1$ (otherwise we would have to replace r by $r' = \frac{\alpha r}{v}$).

B.1 Equations

The continuation value of winning is $v^{2z} = 1$, and of losing is $v^{y2} = 0$, for any $y, z \in \{0, 1\}$. The continuation value in other states are given by the system of equations described in Section 2.2: the 4 unknown variables $\{v^{yz} : y, z \in \{0, 1\}\}$ are given by the system of 4 equations

$$0 = \tfrac{1}{2}(v^{y+1,z} - v^{yz})^2 - e^{zy}(v^{y,z+1} - v^{yz}) - rv^{yz}, \quad y, z \in \{0, 1\} \tag{13}$$

where $e^{zy} = v^{z+1,y} - v^{zy} > 0$ represents the effort of the rival.

Lemma 12. *The inequalities $v^{y,z+1} < v^{yz} < v^{y+1,z}$ hold for any $y, z \in \{0, 1\}$.*

Proof. First, $v^{y+1,z} > v^{yz}$ holds trivially as $e^{yz} > 0$ by assumption. The inequality $v^{y,z+1} < v^{yz}$ holds trivially for $z = 1$ as $v^{y,2} = 0$, it remains to prove it for $z = 0$. We will use mathematical induction, in which we show that weak inequality $v^{y+1,1} \leq v^{y+1,0}$ implies the strong inequality $v^{y,1} < v^{y,0}$. We have $v^{21} \leq v^{20}$ as both values are equal to 1. Consider

[18] The notation $f(1_-)$ stands for $\lim_{x \nearrow 1} f(x)$.

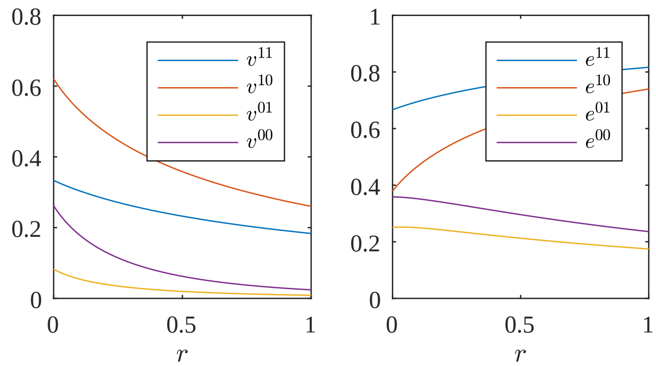

Figure 13: Continuation values and efforts in the four different states of the public information version of the game as a function of the discount factor (research difficulty) r.

$y \in \{0,1\}$, and assume that the inequality $v^{y+1,1} \leq v^{y+1,0}$ holds. Then

$$\begin{aligned}
0 &= \max_{e\in[0,1]} \left\{ e \cdot v^{y+1,0} - \tfrac{1}{2}e^2 + e^{0y} \cdot (v^{y1} - v^{y0}) - (r+e) \cdot v^{y0} \right\} \\
&\geq e^{y1} \cdot v^{y+1,0} - \tfrac{1}{2}(e^{y1})^2 + e^{0y} \cdot (v^{y1} - v^{y0}) - (r+e^{y1}) \cdot v^{y0} \\
&\geq e^{y1} \cdot v^{y+1,1} - \tfrac{1}{2}(e^{y1})^2 + e^{0y} \cdot (v^{y1} - v^{y0}) - (r+e^{y1}) \cdot v^{y0} \\
&= e^{y1} \cdot v^{y+1,1} - \tfrac{1}{2}(e^{y1})^2 - (r+e^{y1}) \cdot v^{y1} + (r + e^{y1} + e^{0y}) \cdot (v^{y1} - v^{y0}) \\
&> \underbrace{e^{y1} \cdot v^{y+1,1} - \tfrac{1}{2}(e^{y1})^2 + e^{1y} \cdot (v^{y2} - v^{y1}) - (r+e^{y1}) \cdot v^{y1}}_{=0} + (r + e^{y1} + e^{0y}) \cdot (v^{y1} - v^{y0}) \\
&= (r + e^{y1} + e^{0y}) \cdot (v^{y1} - v^{y0}),
\end{aligned}$$

and so $0 > v^{y1} - v^{y0}$. We conclude that $v^{y1} < v^{y0}$ for $y \in \{0,1\}$. □

B.2 Uniqueness

Lemma 13. *The system of 4 equations (13) has a unique solution* $(v^{yz} : y, z \in \{0,1\}) \in [0,1]^4$.

Proof of Lemma 13. We prove it recursively by decreasing y and z. For $y = 2$ or $z = 2$ the uniqueness is trivial. Take any $y, z \in \{0,1\}$ for which the uniqueness of $v^{y+1,z}, v^{y,z+1}, v^{z+1,y}, v^{z,y+1}$ has been proven already (initially it is the case for $y = z = 1$). Separating $e^{zy} = v^{z+1,y} - v^{zy}$ in the equation (13),

$$v^{zy} = v^{z+1,y} - e^{zy} = v^{z+1,y} - \frac{\frac{1}{2}(v^{y+1,z} - v^{yz})^2 - rv^{yz}}{v^{yz} - v^{y,z+1}} =: g^{zy}(v^{yz}),$$

so that v^{zy} is expressed as a function of v^{yz} and other variables which are already known to be uniquely defined. Notice that since $v^{y,z+1} \leq v^{y+1,z+1} < v^{y+1,z}$ (Lemma 12), we have

$$\tfrac{1}{2}(v^{y+1,z} - v^{y,z+1})^2 - rv^{y,z+1} > \tfrac{1}{2}(v^{y+1,z+1} - v^{y,z+1})^2 - rv^{y,z+1} \geq 0,$$

and so the strictly decreasing function $x \mapsto \frac{1}{2}(v^{y+1,z} - x)^2 - rx$ has a unique root on the interval $(v^{y,z+1}, v^{y+1,z})$, denote it $\overline{v}^{yz}$. Then

$$g^{zy}(x) = v^{z+1,y} - \frac{\frac{1}{2}(v^{y+1,z} - x)^2 - rx}{x - v^{y,z+1}}$$

is a strictly increasing function on the interval $(v^{y,z+1}, \overline{v}^{yz}]$ and since

$$\begin{aligned} g^{zy}(x) &= v^{z+1,y} - \frac{\frac{1}{2}x^2 - v^{y+1,z}x + \frac{1}{2}(v^{y+1,z})^2 - rx}{x - v^{y,z+1}} \\ &= v^{z+1,y} - \frac{1}{2}v^{y,z+1} + v^{y+1,z} + r - \frac{1}{2}x - \frac{\frac{1}{2}(v^{y+1,z} - v^{y,z+1})^2 - rv^{y,z+1}}{x - v^{y,z+1}}, \end{aligned}$$

it can be written as $g^{zy}(x) = a - \frac{1}{2}x - \frac{c}{x - v^{y,z+1}}$ with $c > 0$, which means that $g^{zy}(x)$ is concave. In summary, $g^{zy}(x)$ is a continuous, concave, strictly increasing function on the interval $(v^{y,z+1}, \overline{v}^{yz}]$ with range from $-\infty$ to $v^{z+1,y}$.

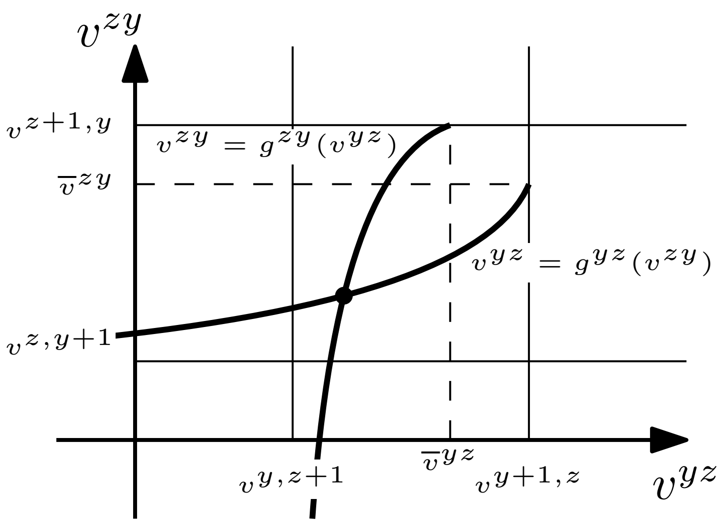

Figure 14: Illustration of the uniqueness of v^{yz} and v^{zy} as an intersection of the graphs of reaction functions.

By symmetry, there is continuous, concave, strictly increasing function $g^{yz}(x)$ defined on the interval $(v^{z,y+1}, \overline{v}^{zy}]$ with range from $-\infty$ to $v^{y+1,z}$, such that $v^{yz} = g^{yz}(v^{zy})$. As illustrated in Figure 14 it should be clear that there is unique point $(v^{yz}, v^{zy}) \in (v^{y,z+1}, \overline{v}^{yz}] \times (v^{z,y+1}, \overline{v}^{zy}]$ that satisfies both $v^{zy} = g^{zy}(v^{yz})$ and $v^{yz} = g^{yz}(v^{zy})$.

□

B.3 Efforts

Lemma 14. *Let*

$$\begin{aligned} h^{11}(x^{11}) &:= \tfrac{3}{2}(x^{11})^2 + (r-1)x^{11} - r \\ h^{10}(x^{10}) &:= \tfrac{1}{2}(x^{10})^2 + (r+e^{01})x^{10} - e^{01}e^{11} - r \\ h^{01}(x^{01}) &:= \tfrac{1}{2}(x^{01})^2 + (r+e^{10})x^{01} - (r+e^{10})(1-e^{11}) \\ h^{00}(x^{00}) &:= \tfrac{3}{2}(x^{00})^2 + x^{00}(r-e^{11}+e^{10}-e^{01}) - r(1-e^{10}). \end{aligned}$$

Then $x > e^{yz}$ ($x < e^{yz}$) whenever $h^{yz}(x) > 0$ ($h^{yz}(x) < 0$), for any $x \geq 0$ and $y, z \in \{0, 1\}$.

Proof. Equation (13) can be written as $g^{yz}(e^{yz}) = 0$, for all $y, z \in 0, 1$. For any $y, z \in 0, 1$, the quadratic polynomial g^{yz} has a positive leading coefficient and a negative intercept, and so it has a unique positive root. Hence, e^{yz} must be the root, and g^{yz} is negative left from it, and positive right from it. □

Lemma 15. *The following estimates hold:*

(i) $\underline{e}^{11} := 1 - \frac{1}{3+2r} < e^{11} < 1 - \frac{1}{4+2r} =: \overline{e}^{11}$;

(ii) $e^{01} < 1 - e^{11} <= \frac{1}{3+2r} =: \overline{e}^{01}$;

(iii) $e^{10} < 1 - \frac{1}{2+2r} =: \overline{e}^{10}$;

(iv) $\underline{e}^{00} := \frac{1}{3+2r} < e^{00}$.

Proof. (i) Evaluating h^{11} at the lower and upper estimate of e^{11},

$$h^{11}(\underline{e}^{11}) = -\frac{r}{(2r+3)^2} < 0, \quad \text{and} \quad g^{11}(\overline{e}^{11}) = \frac{3}{8(r+2)^2} > 0.$$

(ii) Applying the result of (i), $e^{01} = v^{11} - v^{01} < v^{11} = 1 - e^{11} < 1 - \underline{e}^{11} = \frac{1}{3+2r}$.

(iii) We have

$$h^{10}(\overline{e}^{10}) = \tfrac{1}{2}(\overline{e}^{10})^2 + r\overline{e}^{10} + e^{01}(\overline{e}^{10} - e^{11}) - r.$$

Notice that

$$\overline{e}^{10} - \overline{e}^{11} = \frac{1}{(2+2r)(2+r)} < 0,$$

and thus

$$e^{01}(\overline{e}^{10} - e^{11}) > e^{01}(\overline{e}^{10} - \overline{e}^{11}) > \overline{e}^{01}(\overline{e}^{10} - \overline{e}^{11}).$$

Consequently,

$$\begin{aligned} h^{10}(\overline{e}^{10}) &> \tfrac{1}{2}(\overline{e}^{10})^2 + r\overline{e}^{10} + \overline{e}^{01}(\overline{e}^{10} - \overline{e}^{11}) - r \\ &= \frac{2+3r+2r^2}{2(2+2r)^2(2r^2+7r+6)} > 0, \end{aligned}$$

and thus $\overline{e}^{10} > e^{10}$.

(iv) We have

$$\begin{aligned} h^{00}(\underline{e}^{00}) &= \tfrac{1}{2}(\underline{e}^{00})^2 + \underline{e}^{00}(r - e^{11} + e^{10}) - r(1 - e^{10}) \\ &< \tfrac{1}{2}(\underline{e}^{00})^2 + \underline{e}^{00}(r - \underline{e}^{11} + \overline{e}^{10}) - r(1 - \overline{e}^{10}) \\ &= -\frac{r}{(3+2r)^2} < 0 \end{aligned}$$

□

Proof of Proposition 1. To begin with, by Lemma 13 the system of equations (13) has a unique solution, which allows us to analyze it. The inequality $e^{10} < e^{11}$ follows from the fact that $e^{1z} = 1 - v^{1z}$, $z = 0, 1$, and the inequality $v^{11} < v^{10}$ (Lemma 12). The inequality $e^{01} < e^{00}$ follows from the estimates in Lemma 15 as $e^{01} < \overline{e}^{01} = \underline{e}^{00} < e^{00}$. □

C Patent Race with Private Information

The system of ODEs (2) can be expressed in terms of optimal effort levels, as any $\{(e^1_t, e^0_t, p_t), t \geq 0\}$ that satisfy

$$\begin{aligned} \dot{e}^1_t &= \tfrac{1}{2}(e^1_t)^2 - (r + p_t e^1_t)(1 - e^1_t) \\ \dot{e}^0_t &= \tfrac{1}{2}(e^0_t)^2 - \tfrac{1}{2}(e^1_t)^2 + (r + p_t e^1_t)e^0_t \\ \dot{p}_t &= (1 - p_t)(e^0_t - p_t e^1_t), \end{aligned} \tag{14}$$

with the initial condition $p_0 = \hat{p} \in [0, 1)$, and restrictions $e^1_t, p_t \in [0, 1], e^0_t \in [0, 1 - e^1_t]$ for all $t \geq 0$.

Note that the vector $(\dot{e}^1, \dot{e}^0, \dot{p})$ is a continuous function of the vector (e^1, e^0, p), and thus the solution vector is an analytic function of time.

The proof of the Proposition 3 relies on the following three lemmas:

Lemma 16. *The system of ODEs (14) has a unique critical point (e^1_*, e^0_*, p_*) with $p_* < 1$.*

Lemma 17. *The Jordan matrix of the system of ODEs (14) at the critical point (e^1_*, e^0_*, p_*) has a unique eigenvalue with a negative real part.*

Lemma 18. *Any solution of the system of ODEs (14) converges to the steady-state (e^1_*, e^0_*, p_*).*

C.1 Unique Critical Point

Proof of Lemma 16. We prove an equivalent statement that the ODE (14) has a unique critical point with $p < 1$. Letting $(\dot{e}^1_t, \dot{e}^0_t, \dot{p}_t) = 0$, and dropping the time index, we obtain $e^0 = pe^1$ and

$$0 = \tfrac{1}{2}(e^1)^2 - (r + e^0)(1 - e^1) \tag{15}$$

$$0 = \tfrac{1}{2}(e^0)^2 - \tfrac{1}{2}(e^1)^2 + (r + e^0)e^0, \tag{16}$$

which is equivalent to[19]

$$\begin{aligned} e^0 &= \frac{(e^1)^2}{2(1-e^1)} - r \\ 1 &= \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1-e^1}. \end{aligned} \tag{17}$$

Define the function g on the interval $(0,1)$ by

$$g(x) = \left(\frac{\phi(x)}{x}\right)^2 + \frac{\phi(x)}{1-x}.$$

Then the initial system of equations is equivalent to $1 = g(e^1), e^0 = \phi(e^1)$, and $p = \frac{e^0}{e^1}$.

Recall that by Lemma 11 the function ϕ is increasing. The function g is strictly increasing on the interval $[\phi^{-1}(0), 1)$, as both $\frac{\phi(e^1)}{1-e^1}$ and $\frac{\phi(e^1)}{e^1}$ are strictly increasing functions of e^1 so long as $\phi(e^1) \geq 0$. Also, $g(\phi^{-1}(0)) = 0$, and

$$g(e^{11}) > \left(\frac{\phi(e^{11})}{e^{11}}\right)^2 = 1.$$

We conclude that there exists a unique $e^1_* \in (\phi^{-1}(0), e^{11})$ such that $g(e^1_*) = 1$. Put $e^0_* = \phi(e^1_*)$ and $p_* = \frac{e^0_*}{e^1_*}$. Applying the function ϕ to the inequality $e^1_* < e^{11}$, we obtain $e^0_* < e^1_*$, which implies $p_* < 1$. The tuple (e^1_*, e^0_*, p_*) is the unique critical point of the system of ODEs (14) with $p_* < 1$. □

Lemma 19. *At the critical point* $(e^1, e^0, p) = (e^1_*, e^0_*, p_*)$*:*

(i) $\frac{e^1}{v^1} > 1$ *and* $e^1 > \frac{1}{2}$*;*

(ii) $p < 2e^0$*;*

(iii) $r + e^0 > \frac{1}{2}e^1$*;*

(iv) $e^0 > v^1 e^1$ *and* $p > v^1$*.*

Proof. (i) Notice that

$$e^0 = \phi(e^1) - r < \phi(e^1) = \frac{(e^1)^2}{2(1-e^1)},$$

and so (17) gives us

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1-e^1} < \left(\frac{\frac{(e^1)^2}{2(1-e^1)}}{e^1}\right)^2 + \frac{\frac{(e^1)^2}{2(1-e^1)}}{1-e^1} = \frac{3}{4}\left(\frac{e^1}{1-e^1}\right)^2.$$

Thus $\frac{e^1}{v^1} = \frac{e^1}{1-e^1} > \frac{2}{\sqrt{3}} > 1$, which implies $e^1 > \frac{1}{2}$.

[19] The equation (17) is obtained by substituting for $(r + e^0)$ in (16) from (15), and dividing the equation by $\frac{1}{2}(e^1)^2 > 0$.

(ii) We have $p = e^0/e^1 < 2e^0$.

(iii) Applying the result of (i),

$$r + e^0 = \frac{(e^1)^2}{2v^1} = \frac{e^1}{v^1} \cdot \tfrac{1}{2}e^1 > \tfrac{1}{2}e^1.$$

(iv) If $e^0 \le e^1 v^1$, then the equation (17) would lead to a contradiction

$$1 \le (v^1)^2 + e^1 < v^1 + e^1 = 1.$$

Thus $e^0 > e^1 v^1$ and so $p = \frac{e^0}{e^1} > v_1$.

□

C.2 The Jacobian

Consider the critical point $(e^1, e^0, p) = (e^1_*, e^0_*, p_*)$ and define $R := r + 2e^0$ and $d := e^1 - e^0$. The following lemma guarantees us that there is a unique direction from which the critical point can be approached.

Lemma 20. *The Jacobian at the critical point of ODE (14) has a unique eigenvalue with a nonpositive real part, in fact it is a strictly negative real number.*

Proof. The Jacobian of the ODE (14) is equal to (recall that $e^0 = pe^1$)

$$J = \begin{bmatrix} R + e^1 - p & 0 & -v^1 e^1 \\ -e^1 + pe^0 & R & e^0 e^1 \\ -p(1-p) & 1-p & -d \end{bmatrix}.$$

Eigenvalues λ of J are the roots of the polynomial $P(\lambda) := |J - \lambda I|$,

$$P(\lambda) = (R + e^1 - p - \lambda)[(R - \lambda)(-d - \lambda) - e^0 d] + v^1 d[e^1 - p(R + e^0 - \lambda)].$$

We can express the polynomial in terms of its coefficients as $P(\lambda) = -\lambda^3 + a_2\lambda^2 - a_1\lambda + a_0$. Then, by Lemma 19 (ii),$p < 2e^0$, and so

$$a_2 = 2r + 5e^0 - p > 2r + 3e^0 > 0.$$

Next, using inequalities from Lemma 19 (i) and (iv),

$$\begin{aligned} a_0/d = P(0)/d &= -(R + e^1 - p)(R + e^0) + e^1 v^1 - p(R + e^0)v^1 \\ &= -[R + e^1 - p(1 - v^1)](R + e^0) + e^1 v^1 \\ &= -(r + e^0 + e^1)(r + 3e^0) + e^1 v^1 \\ &< -\tfrac{1}{2}(r + 3e^0) + e^0 < 0. \end{aligned}$$

Since $P(0) < 0$ and $\lim_{\lambda \to -\infty} P(\lambda) = +\infty$, the polynomial P needs to have at least one negative root. Denote it μ_1, it remains to prove that the other two roots μ_2, μ_3 have positive real parts. According to Viète's formulas, $a_2 = \mu_1 + \mu_2 + \mu_3$ and $a_0 = \mu_1\mu_2\mu_3$. Then $\mu_2 + \mu_3 = a_2 - \mu_1 > 0$. If the roots μ_2 and μ_3 are real numbers, then at least one of them is positive, and hence both of them have to be positive as $\mu_2\mu_3 = \frac{a_0}{\mu_1} > 0$. Finally, if the roots have a nonzero imaginary parts, then they must be complex conjugates of each other, and thus have positive real parts. □

Lemma 21. *The unique negative eigenvalue of the Jacobian J satisfies the inequality $-\lambda_1 > e^1 - e^0$.*

Proof. We have

$$P(-d) = -(r + e^0 + 2e^1 - p)e^0 d + v^1 d[e^1 - p(r + 2e^0 + e^1)].$$

By Lemma 19 (i) and (ii), $e^1 > v^1$ and $p < 2e^0$, and so $(r + e^0 + 2e^1 - p)e^1 > rv^1$. Thus,

$$\begin{aligned} P(-d)/(pdv^1) &< -r + e^1/p - r - 2e^0 - e^1 \\ &= e^1/p - e^1 - 2(r + e^0) \\ &= \tfrac{1}{e^0}[(e^1)^2 - e^1e^0 - 2(r + e^0)e^0] \\ &< \tfrac{1}{e^0}[(e^1)^2 - (e^0)^2 - 2(r + e^0)e^0] = 0, \end{aligned}$$

where we used the equation (16). Since the polynomial P has a unique negative root, $P(-d) < 0$, and $\lim_{x\to-\infty} P(x) = +\infty$, necessarily $\lambda_1 < -d$. □

Lemma 22. *The eigenvector $\mu = (\mu_{e^1}, \mu_{e^0}, \mu_p)$ of the Jordan matrix J associated with the negative eigenvalue λ_1 is such that $\frac{\mu_{e1}}{\mu_p} > 0$ and $\frac{\mu_{e0}}{\mu_p} < 0$.*

Proof. The eigenvector associated with λ_1 is characterized by the equation $(J - \lambda_1 I)\mu = 0$, which is equivalent to

$$\begin{aligned} (R + e^1 - p - \lambda_1)\mu_{e^1} - v^1e^1\mu_p &= 0, \\ -(e^1 - pe^0)\mu_{e^1} + (R - \lambda_1)\mu_{e^0} + e^0e^1\mu_p &= 0. \end{aligned}$$

Clearly, $\mu_p \neq 0$, as otherwise the whole eigenvector μ would be zero. Since the coefficient of μ_{e^1} in the first equation is positive, $\frac{\mu_{e1}}{\mu_p} > 0$. Substituting for μ_{e^1} from the first into the second equation and using the inequality $-\lambda_1 > e^1 - e^0$ (Lemma 21) together with the inequality $e^0 > e^1v^1$ and $e^1 > \frac{1}{2}$ (Lemma 19),

$$\begin{aligned} &\frac{1}{e^1}(R + e^1 - p - \lambda_1)(R - \lambda_1)\frac{\mu_{e^0}}{\mu_p} \\ &\quad = (e^1 - pe^0)v^1 - e^0(R + e^1 - p - \lambda_1) \\ &\quad = e^1v^1 - e^0(R + e^1 - pe^1 - \lambda_1) \\ &\quad = e^1v^1 - e^0(r + e^0 + e^1 - \lambda_1) \\ &\quad < e^0 - e^0(r + 2e^1) < 0. \end{aligned}$$

In conclusion, $\frac{\mu_{e0}}{\mu_p} < 0$. □

C.3 Convergence to the Critical Point

The following lemma will help us to prove that if p_t converges monotonically, then the whole solution vector has to converge.[20]

Lemma 23. *Let $F_t : [0,1] \to \mathbb{R}, t \in \mathbb{R}_+$ be a system of continuously differentiable functions that uniformly converge to some continuous function $F_\star[0,1] \to \mathbb{R}$ as $t \to \infty$ that has a unique root $x_\star$. Assume that $F'_\star(x_\star) > 0$. If $\dot{x}_t = F_t(x_t)\ \forall t \geq 0$, then $x_t \to x_\star$ as $t \to \infty$.*

Proof. Consider a fixed $\varepsilon > 0$. Define $L_\varepsilon = \min\{|F_\star(x)|, x \in [0,1], |x - x_\star| \geq \varepsilon\}$. Since $F_\star$ is continuous with unique root $x_\star$, L_ε is well defined and positive.

Since the functions F_t converge uniformly to $F_\star$, there is $T_\varepsilon \geq 0$ such that $|F_t(x) - F_\star(x)| < \frac{1}{2}L_\varepsilon$ for all $x \in [0,1]$ and $t \geq T_\varepsilon$. Then, $|F_t(x) - F_\infty(x| < \frac{1}{2}L_\varepsilon$, for all $t \geq T_\varepsilon$ and $x \in [0,1]$.

Using a triangle inequality, we conclude that if $|x_t - x_\star| \geq \varepsilon$ and $t \geq T_\varepsilon$, then

$$\begin{aligned} |F_t(x_t)| &= |F_t(x_t) - F_\star(x_\infty)| \\ &\geq |F_\star(x_t) - F_\star(x_\infty)| - |F_t(x_t) - F_t(x_t)| \\ &\geq L_\varepsilon - \tfrac{1}{2}L_\varepsilon = \tfrac{1}{2}L_\varepsilon. \end{aligned}$$

It follows that if $x_{t_0} \leq x_\star - \varepsilon$ (alternatively $x_{t_0} \geq x_\star + \varepsilon$) for some $t_0 \geq T_\varepsilon$, then $x_t \leq x_\star - \varepsilon - \frac{1}{2}L_\varepsilon(t - t_0)$ (alternatively $x_{t_0} \geq x_\star + \varepsilon + \frac{1}{2}L_\varepsilon(t - t_0)$) for all $t \geq t_0$, and eventually x_t goes out of bounds. We conclude that for every $\varepsilon > 0$ and every $t \geq T_\varepsilon$, $|x_t - x_\star| < \varepsilon$, and so $x_t \to x_\star$ as $t \to \infty$. □

Lemma 24. *If $p_t < 1$ for all $t \geq 0$ and $p_t \to p_\star \in [0,1]$, then $(e^1_t, e^0_t, p_t) \to (e^1_*, e^0_*, p_*)$ as $t \to +\infty$.*

Proof. The ODE (14) for e^1_t can be written as $\dot{e}^1_t = F^1_t(e^1_t)$, where

$$F^1_t(x) := \tfrac{1}{2}(x)^2 - (r + p_t x)(1 - x)$$

are continuously differentiable functions converging uniformly to $F^1_\star(x) = \frac{1}{2}x^2 - (r + p_\star x)(1 - x)$ as $t \to \infty$. Since $F^1_\star$ is a quadratic function with a positive leading coefficient, negative intercept, and $F^1(1) > 0$, it has a unique positive root $e^1_\star \in (0,1)$ and $(F^1_\star)'(e^1_\star) > 0$. Applying Lemma 23, we conclude that $e^1_t \to e^1_\star$.

Similarly, the ODE (14) for e^0_t can be written as $\dot{e}^0_t = F^0_t(e^0_t)$, where

$$F^0_t(x) := \tfrac{1}{2}x^2 - \tfrac{1}{2}(e^1_t)^2 + (r + p_t e^1_t)x$$

are continuously differentiable functions converging uniformly to the function $F^0_\star(x) = \frac{1}{2}x^2 - \frac{1}{2}(e^1_\star)^2 + (r + p_\star e^1_\star)x$ as $t \to \infty$. Since F^0_∞ is a quadratic polynomial with a positive leading coefficient, negative intercept, and $F^0_\star(e^1_\star) > 0$, it has a unique positive root $e^0_\star \in (0, e^1_\star)$, and $(F^0_\star)'(e^0_\star) > 0$. Applying Lemma 23, we conclude that $e^0_t \to e^0_\star$.

Since $e^0_\star < e^1_\star$, it must be that $p_\star \leq 1$, because otherwise $\dot{p}_t$ would necessarily be negative for t large (which would prevent it from approaching to 1). We conclude that $(e^1_\star, e^0_\star, p_\star)$ is a critical point of the ODE (14) with $p_\star < 1$, and thus, by Lemma 16, $(e^1_\star, e^0_\star, p_\star) = (e^1_*, e^0_*, p_*)$. □

[20] We use index '$\star$' for a general limit, whilst the index '$*$' we keep reserved for the critical point.

Proof of Lemma 18. In any Markov perfect Bayesian equilibrium, p_t must be monotonous as otherwise there would be $0 < t_1 < t_2$ such that $p_{t_1} = p_{t_2}$, but $\dot{p}_{t_1} \neq \dot{p}_{t_2}$, which is not consistent with the Markov property.[21]

Since p_t is monotonous on a bounded range, it has to converge to some value $p_\star$. The rest follows from the Lemma 24. □

C.4 Effort Over Time

Suppose that there exists a solution of the system of ODEs (14) on some interval $p \in [\underline{p}, \overline{p}]$. Then it has the following properties:

Lemma 25. *For any $p \in [0,1]$, $\frac{\partial}{\partial p} e^1(p) > 0$.*

Lemma 26. *For any $p \in [0,1)$, $\frac{\partial}{\partial p} e^0(p) < 0$.*

Lemma 27. *The effort of a player is higher if he is successful, i.e. $e^1(p) > e^0(p)$ for any $p \in [0,1]$.*

Proof of Lemma 25. By Lemma 22, the direction $(\nu^{e1}, \nu^{e0}, \nu^p)$ in which the solution has to converge to the steady-state is such that $\frac{\nu^{e1}}{\nu^p} > 0$, and so the claim holds at $p = p^*$. For contradiction, suppose that the claim is violated at some $p^1 \in [0,1]$. Then there exists $p^0 \in [0,1]$ such that $\frac{\partial}{\partial p} e^1(p_0) = 0$. Consider such p^0 that is closest to p^*. At such point $p_t = p^0$ implies $\ddot{e}^1_t \geq [\leq] 0$ whenever $p^0 < [>] p^*$. However, taking the derivative of the formula

$$\dot{e}^1_t = \tfrac{1}{2}(e^1_t)^2 - (r + p_t e^1_t)(1 - e^1_t),$$

we obtain (substituting $\dot{e}^1(p^0) = 0$)

$$\ddot{e}^1_{t_1} = -e^1_{t_1}(1 - e^1_{t_1})\dot{p}_{t_1} < [>] 0,$$

whenever $p^0 < [>] p^*$ (Lemma 18). Contradiction. □

Proof of Lemma 26. By Lemma 22, the direction $(\nu^{e1}, \nu^{e0}, \nu^p)$ in which the solution has to converge to the steady-state is such that $\frac{\nu^{e0}}{\nu^p} < 0$, and so the claim holds at $p = p^*$. For contradiction, suppose there is $p^1 \in [0,1]$ such that $\dot{e}^0(p^0) \geq 0$. Then there exists $p^0 \in [0,1]$ such that $\frac{\partial}{\partial p} e^0(p_0) = 0$. Consider such p^0 that is closest to p^*. At such point $p_t = p^0$ implies $\ddot{e}^1_t \leq [\geq] 0$ whenever $p^0 < [>] p^*$. However, taking the derivative of the formula

$$\dot{e}^0_t = \tfrac{1}{2}(e^0_t)^2 - \tfrac{1}{2}(e^1_t)^2 + (r + p_t e^1_t) e^0_t,$$

we obtain

$$\ddot{e}^0_{t_1} = e^1_t(-\dot{e}^1_t + e^0_t \dot{p}_t) + p_t e^0_t \dot{e}^1_t > [<] 0,$$

whenever $p^0 < [>] p^*$ (Lemma 18). Contradiction. □

[21] Indeed, p_t is the only state in the game, so if p_t is the same at the two times, then also v^1_t, v^0_t has to be the same. But that implies that also $\dot{p}_t$ is the same at the two times.

Proof of Lemma 27. By Lemma 26, $p_t < p^*$ implies

$$0 > \dot{e}_t^0 = \tfrac{1}{2}(e_t^0)^2 - \tfrac{1}{2}(e_t^1)^2 + (r + p_t e_t^1)e_t^0,$$

thus necessarily $e_t^1 > e_t^0$. For $p \geq p^*$ the claim then follows by Lemma 25 and Lemma 26. $\square$

C.5 Existence of a Unique Solution

Proof of Proposition 3. From Lemma 16, 18, and 17 we know that every solution of the ODE (2) has to converge to a unique critical point from a uniquely direction. The functions $e^0(p)$ and $e^1(p)$, $p \in [0,1)$ are characterized by the following equations

$$\frac{\partial}{\partial p}e^1 = \frac{\frac{1}{2}(e^1)^2 - (r + pe^1)(1 - e^1)}{(1-p)(e^0 - pe^1)} \tag{18}$$

$$\frac{\partial}{\partial p}e^0 = \frac{\frac{1}{2}(e^0)^2 - \frac{1}{2}(e^1)^2 + (r + pe^1)e^0}{(1-p)(e^0 - pe^1)}, \tag{19}$$

for $p \in [0,1) \setminus \{p_*\}$, and $\frac{\partial}{\partial p}e^1(p_*) = \frac{\mu_{e1}}{\mu_p}$, $\frac{\partial}{\partial p}e^0(p_*) = \frac{\mu_{e0}}{\mu_p}$, $e^1(p_*) = e^1_*$, and $e^0(p_*) = e^0_*$. This system of equations is an initial value problem, so the existence and uniqueness of its solution is guaranteed so long as the derivatives are bounded. Let $\underline{p} \in [0, p_*]$ be minimal such that a solution of the system of ODEs (18)-(19) exists on the interval $[\underline{p}, p_*]$.[22]

Claim that $\underline{p} = 0$. For contradiction, suppose that $\underline{p} > 0$. The nominators of the RHSs of equations (18) and (19) are both bounded, and their denominator is

$$D(p) := (1-p)(e^0(p) - pe^1(p)).$$

From Lemma 22, it follows that

$$\begin{aligned}\frac{\partial}{\partial p}D(p_*) &= (1-p_*)(\frac{\partial}{\partial p}e^0(p_*) - e^1(p_*) - p_*\frac{\partial}{\partial p}e^1(p_*)) \\ &= (1-p_*)(\frac{\mu_{e0}}{\mu_p} - e^1_* - p_*\frac{\mu_{e1}}{\mu_p}) < 0,\end{aligned}$$

so $D(p) > 0$ for $p < p_*$ close enough to p_*, and so the initial value problem has a unique solution close to p_*. Thus $\underline{p} < p^*$. From Lemma 25 and Lemma 26 it follows that $D(p)$ is decreasing, and thus it cannot be the case that $D(\underline{p}) = 0$. The only other thing that could possibly go wrong is if one of the inequalities $0 < e^1(\underline{p}), 0 < e^0(\underline{p})$ or $e^0(\underline{p}) + e^1(\underline{p}) < 1$ was violated. However, neither of those can happen as $0 = e^1(\underline{p})$ implies $\frac{\partial}{\partial p}e^1(\underline{p}) < 0$, which would mean that the inequality would already be violated for $p > \underline{p}$, and analogously for the other inequalities. Nothing could have gone wrong at $\underline{p}$, which is a contradiction with the assumption $\underline{p} > 0$.

We conclude that the initial problem has a unique solution on the interval $[0, p_*]$. Showing that there is unique solution on the interval $[p_*, 1)$ is analogous. $\square$

[22] The existence of the minimum is guaranteed. Let $\underline{p}$ be the infimum. Then the solution exists on the interval $(\underline{p}, p_*]$, and so it can be continuously extended to the interval $[\underline{p}, p_*]$.

D One Player Known to be Successful

Instead of characterizing the trajectory of the vector $(v_t^{1A}, v_t^{1B}, v_t^{0A}, p_t^B)$ by the ODE (3), we can equivalently characterize the trajectory of the vector $(e_t^{1A}, e_t^{1B}, e_t^{0A}, p_t^B)$ by the following ODE:

$$\begin{aligned}
\dot{e}_t^{1A} &= \tfrac{1}{2}(e_t^{1A})^2 - (r + p_t^B e_t^{1B})(1 - e_t^{1A}) \\
\dot{e}_t^{1B} &= \tfrac{1}{2}(e_t^{1B})^2 - (r + e_t^{1A})(1 - e_t^{1B}) \\
\dot{e}_t^{0B} &= \tfrac{1}{2}(e_t^{0B})^2 - \tfrac{1}{2}(e_t^{1B})^2 + (r + e_t^{1A})e_t^{0B} \\
\dot{p}_t^B &= (1 - p_t^B)(e_t^{0B} - p_t^B e_t^{1B}),
\end{aligned} \tag{20}$$

where $p_0^B = \hat{p}^B$, and $e_t^{1A}, e_t^{1B} \in (0, 1]$, $e_t^{1B} \in [0, 1 - e_t^{1B}]$, and $p_t \in [0, 1]$.

Similarly as in the case with both players starting from the state 0, the proof of the existence and uniqueness of the solution consists of three steps: ensuring that the ODE has a unique critical point; that every solution has to converge to it; and that there is unique direction in which it can occur.

Lemma 28. *The ODE (20) has a unique critical point $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ with $p_*^B < 1$.*

Lemma 29. *At the critical point $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$, the Jordan matrix has a unique negative eigenvalue.*

Lemma 30. *Any solution vector $(e_t^{1A}, e_t^{1B}, e_t^{0B}, p_t^B)$ of the ODE (20) with $p_0^B < 1$ converges to the critical point $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ as $t \to +\infty$.*

D.1 Unique Critical Point

Proof of Lemma 28. Similarly as in the proof of Lemma 16, we obtain equations

$$\begin{aligned}
e^{0B} &= \frac{(e^{1A})^2}{2(1 - e^{1A})} - r \\
e^{1A} &= \frac{(e^{1B})^2}{2(1 - e^{1B})} - r \\
1 &= \left(\frac{e^{0B}}{e^{1B}}\right)^2 + \frac{e^{0B}}{1 - e^{1B}}.
\end{aligned} \tag{21}$$

Define the function $h(x)$ on the interval $[0, 1)$ by

$$h(x) := \left(\frac{\phi(\phi(x))}{x}\right)^2 + \frac{\phi(\phi(x))}{1 - x}.$$

A tuple $(e^{1A}, e^{1B}, e^{0B}, p^B)$ is then a critical point of the system of ODEs (20) if and only if $e^{0B} = \phi(e^{1A})$, $e^{1A} = \phi(e^{1B})$, $1 = h(e^{1B})$, and $p = \frac{e^{0B}}{e^{1B}}$.

Consider $x \in [\phi^{-1}(\phi^{-1}(0)), 1)$. Clearly, the functions $\frac{\phi(x)}{2(1-\phi(x))}$ and $\frac{\phi(x)}{x}$ are strictly increasing and positive, and so their product

$$\frac{\phi(\phi(x))}{x} = \frac{(\phi(x))^2}{2(1-\phi(x))x} = \frac{\phi(x)}{2(1-\phi(x))} \cdot \frac{\phi(x)}{x}.$$

It turns out that the function $h(x)$ is a sum of two strictly increasing functions, and so it is strictly increasing itself. Finally, $h(\phi^{-1}(\phi^{-1}(0))) = 0$ and

$$h(e^{11}) > \left(\frac{\phi(\phi(e^{11}))}{e^{11}}\right)^2 = \left(\frac{e^{11}}{e^{11}}\right)^2 = 1,$$

and so $e_*^{1B} = h^{-1}(1) \in (\phi^{-1}(\phi^{-1}(0)), e^{11})$ is unique. Then $e_*^{1A} = \phi(e_*^{1B}) < e_*^{1B}$, $e_*^{0B} = \phi(e_*^{1A}) < e_*^{1A}$, and thus $p_* = \frac{e_*^{0B}}{e_*^{1B}} < 1$. The tuple $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ is the unique critical point of the system of ODEs (20). Note that the condition $e_*^{0B} < 1 - e_*^{1B}$ follows from the equation (21). □

Lemma 31. *At the critical point* $(e^{1A}, e^{1B}, e^{0B}, p^B) = (e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$*:*

(i) $v^{1B} < v^{1A} < \frac{1}{2} < e^{1A} < e^{1B}$*;*

(ii) $e^{0B} > e^{1B}v^{1B}$*.*

Proof. (i) The inequality $e^{1A} < e^{1B}$ has already been proved, and thus it remains to show that $\frac{1}{2} < e^{1A}$. (The rest follows from that $v^{1A} = 1 - e^{1A}$ and $v^{1B} = 1 - e^{1B}$.) Recall that $e^{1B} = h^{-1}(1)$, where h is a strictly increasing function, and $e^{1A} = \phi(e^{1B})$, so the claim can be equivalently put as $x := \phi^{(-1)}(\frac{1}{2}) < h^{(-1)}(1)$, or $h(x) < 1$. We have $\phi(\phi(x)) = \phi(\frac{1}{2}) = \frac{1}{4} - r$, and $x < \frac{1}{2}(-1 + \sqrt{5})$, because $\frac{1}{2} = \phi(x) < \frac{1}{2}x^2/(1-x)$, and so $x^2 = 1 - x$. Consequently,

$$h(x) = \frac{(\phi(\frac{1}{2}))^2}{x^2} + \frac{\phi(\frac{1}{2})}{1-x} = \frac{(\phi(\frac{1}{2}))^2}{1-x} + \frac{\phi(\frac{1}{2})}{1-x} < \frac{\frac{1}{16} + \frac{1}{4}}{\frac{1}{2}(3 - \sqrt{5})} = \frac{5}{8(3 - \sqrt{5}))} < 1.$$

1. If $e^{0B} \le e^{1B}v^{1B}$, then equation (21) leads to a contradiction,

$$1 = \left(\frac{e^{0B}}{e^{1B}}\right)^2 + \frac{e^{0B}}{v^{1B}} \le \left(v^{1B}\right)^2 + e^{1B} < v^{1B} + e^{1B} = 1.$$

□

D.2 The Jacobian

Consider the critical point $(e^{1A}, e^{1B}, e^{0B}, p^B) = (e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ and define

$$R^A := r + e^{1A} + e^{0B}, \quad \text{and} \quad d^B := e^{1B} - e^{0B}.$$

The Jordan matrix of the system of ODE (20) at the critical point is

$$J^A = \begin{bmatrix} R^A & -p^B v^{1A} & 0 & -e^{1B}v^{1A} \\ -v^{1B} & R^A + d^B & 0 & 0 \\ e^{0B} & -e^{1B} & R^A & 0 \\ 0 & -p^B(1-p^B) & (1-p^B) & -e^{1B}(1-p^B) \end{bmatrix}$$

The eigenvalues of J^A are the complex roots of the polynomial $P^A(\lambda) = \det(J^A - \lambda I)$, where I is the identity matrix.

Proof of Lemma 29. By definition,

$$P^A(\lambda) = \begin{vmatrix} R^A - \lambda & -p^B v^{1A} & 0 & -e^{1B}v^{1A} \\ -v^{1B} & R^A + d^B - \lambda & 0 & 0 \\ e^{0B} & -e^{1B} & R^A - \lambda & 0 \\ 0 & -p^B(1-p^B) & (1-p^B) & -e^{1B}(1-p^B) - \lambda \end{vmatrix}$$

Subtracting p^B/e^{1B} times the last column of the determinant from its second column, and using the identity $e^{1B}(1-p^B) = d^B$,

$$P^A(\lambda) = \begin{vmatrix} R^A - \lambda & 0 & 0 & -e^{1B}v^{1A} \\ -v^{1B} & R^A + d^B - \lambda & 0 & 0 \\ e^{0B} & -e^{1B} & R^A - \lambda & 0 \\ 0 & \frac{p^B}{e^{1B}}\lambda & 1-p^B & -d^B - \lambda \end{vmatrix}.$$

Expanding the determinant by the first row,

$$\begin{aligned} P^A(\lambda) = {} & (R^A - \lambda)^2(R^A + d^B - \lambda)(-d^B - \lambda) \\ & + e^{1B}v^{1A} \begin{vmatrix} -v^{1B} & R^A + d - \lambda & 0 \\ e^{0B} & -e^{1B} & R^A - \lambda \\ 0 & \frac{p^B}{e^{1B}}\lambda & 1 - p^B \end{vmatrix}, \end{aligned}$$

in which the 3×3 determinant equals to

$$-v^{1B} \begin{vmatrix} -e^{1B} & R^A - \lambda \\ \frac{p^B}{e^{1B}}\lambda & 1 - p^B \end{vmatrix} - (R^A + d^B - \lambda) \begin{vmatrix} e^{0B} & R^A - \lambda \\ 0 & 1 - p^B \end{vmatrix}.$$

Consequently,

$$\begin{aligned} P^A(\lambda) = {} & (R^A - \lambda)^2(R^A + d^B - \lambda)(-d^B - \lambda) \\ & + v^{1A}v^{1B}[e^{1B}d^B + p^B(R^A - \lambda)\lambda)] - v^{1A}e^{0B}d^B(R^A + d^B - \lambda). \end{aligned}$$

We can express the polynomial in terms of its coefficients as $P^A(\lambda) = \lambda^4 - b_3\lambda^3 + b_2\lambda^2 - b_1\lambda + b_0$. Then, taking into account the inequalities $e^{1A} > \frac{1}{2} > v^{1A}$ and $e^{1B} > \frac{1}{2} > v^{1B}$ (Lemma 31),

$$\begin{aligned} b_0/d^B = P(0)/d^B &= -(R^A)^2(R^A + d^B) + v^{1A}[v^{1B}e^{1B} - e^{0B}(R^A + d^B)] \\ &< -(e^{1A})^2 e^{1B} + v^{1A}v^{1B}e^{1B} < 0. \end{aligned}$$

Thus P^A has at least one positive and one negative root (recall that $P^A(\lambda)$ goes to infinity as λ goes either to positive or to negative infinity). Denote λ_1^A the smallest negative root of P^A, it remains to prove its uniqueness. We have

$$b_3 = 2R^A + (R^A + d^B) - d^B = 3R^A > 0.$$

Since $P^{A(3)}(\lambda) = 24\lambda - 6b_3 < 0$ for all $\lambda \leq 0$, $P^{A(2)}(\lambda)$ is decreasing. In addition to that (recall that R^A is greater than either of d^B, v^{1A}, and v^{1B})

$$P^{A(2)}(0)/2 = b_2 = 2(R^A)^2 - d^B R^A - v^{1A}v^{1B}p^B > 0.$$

It follows that $P^{A(2)}(\lambda) > 0$ for all $\lambda \leq 0$, and so λ_1^A is the unique negative root of P^A as the polynomial is convex on $(-\infty, 0]$.

It remains to exclude the possibility of P^A having a root with a nonpositive real part and a nonzero imaginary part. For contradiction, suppose that λ_2^A was such a root. Then its complex conjugate λ_3^A would also be a root. Denote λ_4^A the positive root of P^A. By Viète's formulas, $b_3 = \lambda_1^A + \lambda_2^A + \lambda_3^A + \lambda_4^A$. Since all the roots except for λ_4^A have a nonpositive real part, $\lambda_4^A > b_3 = 3R^A$. That, however, is in contradiction with the fact that

$$\begin{aligned} P^A(3R^A) = {} & (2R^A)^2(2R^A - d^B)(3R^A + d^B) \\ & + v^{1A}v^{1B}[e^{1B}d^B - 6p^B(R^A)^2] + v^{1A}e^{0B}d^B(2R^A - d^B) > 0. \end{aligned}$$

□

Lemma 32. *The eigenvector $\mu^A = (\mu_{e^{1A}}, \mu_{e^{1B}}, \mu_{e^{0B}}, \mu_{p^B})$ of the Jordan matrix J^A associated with the negative eigenvalue λ_1^A is such that $\frac{\mu_{e1A}}{\mu_{pB}} > 0$, $\frac{\mu_{e1B}}{\mu_{pB}} > 0$ and $\frac{\mu_{e0B}}{\mu_{pB}} < 0$.*

Proof. The eigenvector μ^A is characterized by the vector equation $(J^A - \lambda_1^A I)\mu^A = 0$, which gives us

$$\begin{aligned} (R^A - \lambda_1^A)\mu_{e^{1A}} - p^B v^{1A}\mu_{e^{1B}} - e^{1B}v^{1A}\mu_{p^B} &= 0 \\ -v^{1B}\mu_{e^{1A}} + (R^A + d^B - \lambda_1^A)\mu_{e^{1B}} &= 0 \\ e^{0B}\mu_{e^{1A}} - e^{1B}\mu_{e^{1B}} + (R^A - \lambda_1^A)\mu_{e^{0B}} &= 0. \end{aligned}$$

Substituting from the second equation into the others,

$$\begin{aligned} \left[(R^A - \lambda_1^A)(R^A + d^B - \lambda_1^A) - p^B v^{1A}v^{1B}\right]\mu_{e^{1B}} - e^{1B}v^{1A}v^{1B}\mu_{p^B} &= 0 \\ \left[e^{0B}(R^A + d^B - \lambda_1^A) - e^{1B}v^{1B}\right]\mu_{e^{1B}} + (R^A - \lambda_1^A)v^{1B}\mu_{e^{0B}} &= 0. \end{aligned}$$

Since $R^A > e^{1A} > \frac{1}{2} > v^{1A} > v^{1B}$, $(R^A)^2 > v^{1A}v^{1B}$, and so the coefficient of $\mu_{e^{1B}}$ in the first equation is positive. Consequently, $\frac{\mu_{e1B}}{\mu_{pB}} > 0$, and thus also $\frac{\mu_{e1A}}{\mu_{pB}} > 0$ (clearly, $\mu_{p^B} \neq 0$, as otherwise the whole vector μ^A would be zero). Finally, the coefficient of $\mu_{e^{1B}}$ in the second equation is positive as $e^{0B} > e^{1B}v^{1B}$ (Lemma 31 (ii)), and $R^A + d^B - \lambda_1^A > R^A + d^B = r + e^A + e^B > 1$. Consequently, $\frac{\mu_{e0B}}{\mu_{pB}} < 0$. □

D.3 Convergence to the Critical Point

Lemma 33. *There are strictly increasing functions $E^{1A}, E^{1B} : [0,1] \to [0,1]$ such that:*

(a) *if $p_t^B \geq \underline{p}^B$ for all $t \geq t_0$, then $e_t^{1A} \geq E^{1A}(\underline{p})$ and $e_t^{1B} \geq E^{1B}(\underline{p})$ for all $t \geq t_0$;*

(b) *if $p_t^B \leq \overline{p}^B$ for all $t \geq t_0$, then $e_t^{1A} \leq E^{1A}(\overline{p})$ and $e_t^{1B} \leq E^{1B}(\overline{p})$ for all $t \geq t_0$.*

Proof of Lemma 33. We define the functions E^{1A} and E^{1B} so that $e^{1A} = E^{1A}(p^B)$ and $e^{1B} = E^{1B}(p^B)$ solves the following system of equations:

$$\begin{aligned} 0 &= \tfrac{1}{2}(e^{1A})^2 - (r + p^B e^{1B})(1 - e^{1A}) \\ 0 &= \tfrac{1}{2}(e^{1B})^2 - (r + e^{1A})(1 - e^{1B}). \end{aligned}$$

In other words, the functions are chosen so that $e_t^{1A} = E^{1A}(p_t^B)$ and $e_t^{1B} = E^{1B}(p_t^B)$ would imply $\dot{e}_t^{1A} = 0$ and $\dot{e}_t^{1B} = 0$. This definition is proper. Recall the function $\phi(x) = \frac{x^2}{2(1-x)} - r$ from Lemma 11. The system of equations can be equivalently written as $p^B e^{1B} = \phi(e^{1A})$ and $e^{1A} = \phi(e^{1B})$. Recall that $\phi^{-1} : [0,1] \to [0,1]$ is continuous, strictly increasing and strictly concave function, and so is the function $x \mapsto \phi^{-1}(\phi^{-1}(p^B x))$ then, for any given $p^B \in [0,1]$. Since $\phi^{-1}(\phi^{-1}(p^B \cdot 0)) > 0$ and $\phi^{-1}(\phi^{-1}(p^B \cdot 1)) < 1$, the strictly concave function $x \mapsto \phi^{-1}(\phi^{-1}(p^B x))$ has a unique fixed point on the interval $[0,1]$. It remains to assign the value of this fixed point to e^{1B} and to put $e^{1A} := \phi(e^{1B})$. The functions E^{1A} and E^{1B} are well defined.

The most straightforward way to show that the functions E^{1A} and E^{1B} are strictly increasing is to use (a vector version of) the implicit value theorem. Since the function $x \mapsto \phi^{-1}(\phi^{-1}(p^B x))$ is strictly increasing and strictly concave, it has to have derivative less than 1 right from its fixed point, i.e., for $x \geq E^{1A}(p^B)$. Consequently, for a given $p^B \in [0,1]$, $e_{(n)}^{1A} \searrow E^{1A}(p^B)$, where $e_{(0)}^{1A} = 1$ and $e_{(n)}^{1A} = \phi^{-1}(\phi^{-1}(p^B e_{(n-1)}^{1A}))$, for $n = 1, 2, \ldots$. What is more, we can take the function $e_{(0)}^{1A}(p^B) \equiv 1, p^B \in [0,1]$ and define the sequence of functions $e_{(n)}^{1A}(p^B) = \phi^{-1}(\phi^{-1}(p^B e_{(n-1)}^{1A}(p^B)))$, for $n = 1, 2, \ldots$, that will converge uniformly (monotonic convergence) to the function $E^{1A}(p_B)$. If $e_{(n-1)}^{1A}(p^B)$ is a nondecreasing function of p^B, then $p^B e_{(n-1)}^{1A}(p^B)$ is a strictly increasing function of p^B, and so is $e_{(n)}^{1A}(p^B) = \phi^{-1}(\phi^{-1}(p^B e_{(n-1)}^{1A}(p^B)))$ then as ϕ^{-1} is strictly increasing. As a result, their (uniform) limit, the function $E^{1A}(p^B)$ is an increasing function of p^B. Finally, $E^{1B}(p^B) = \phi^{-1}(p^B e_{(n-1)}^{1A}(p^B))$ is also strictly increasing in p^B then.

We will prove part (a) of the Lemma; the proof of part (b) is analogous. Let there be $\underline{p}^B \in [0,1]$ and $t_0 \geq 0$ such that $p_t^B \geq \underline{p}^B$ for all $t \geq t_0$ and define

$$\underline{e}^{1A} = \inf_{t \geq t_1} e_t^{1A} \quad \text{and} \quad \underline{e}^{1B} = \inf_{t \geq t_1} e_t^{1B}.$$

Then necessarily

$$\begin{aligned} 0 &\leq \tfrac{1}{2}(\underline{e}^{1A})^2 - (r + \underline{p}^B \underline{e}^{1B})(1 - \underline{e}^{1A}) \qquad (22) \\ 0 &\leq \tfrac{1}{2}(\underline{e}^{1B})^2 - (r + \underline{e}^{1A})(1 - \underline{e}^{1B}). \qquad (23) \end{aligned}$$

Indeed, suppose that the first inequality was violated. Then there was $t_1 \geq t_0$ and $\delta > 0$ such that

$$0 > -\delta = \tfrac{1}{2}(e_{t_1}^{1A})^2 - (r + \underline{p}^B \underline{e}^{1B})(1 - e_{t_1}^{1A}) \geq \tfrac{1}{2}(e_{t_1}^{1A})^2 - (r + p_{t_1}^B e_{t_1}^{1B})(1 - e_{t_1}^{1A}) = \dot{e}_{t_1}^{1A},$$

so that e_t^{1A} is falling starting from $t = t_1$ at rate at least δ. Which is a contradiction, because e_t^{1A} is bounded from below by $\underline{e}^{1A}$ and so it can not be falling indefinitely. The second inequality can be proved analogously.

The lowest $\underline{e}^{1A}$ and $\underline{e}^{1B}$ satisfying the inequalities (22) and (23) simultaneously is attained when both constraints are binding (for example, if (22) was slack, then $\underline{e}^{1A}$ could be lowered increasing the slack in the inequality (23)), and that happens when $\underline{e}^{1A} = E^{1A}(\underline{p}^B)$ and $\underline{e}^{1B} = E^{1B}(\underline{p}^B)$. Our claim $e_t^{1A} \geq E^{1A}(\underline{p}^B)$ and $e_t^{1B} \geq E^{1B}(\underline{p}^B)$ follows. □

Lemma 34. *If $p_t^B < 1$ converges, then $(e_t^{1A}, e_t^{1B}, e_t^{0B}, p_t^B) \to (e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ as $t \to +\infty$.*

34. Let $p_t^B \to p_\star^B$. Then p_t^B can be estimated from below and above arbitrarily narrowly for t large, and so by Lemma 33 $e_t^{1A} \to e_\star^{1A} := E^{1A}(p_\star^B)$ and $e_t^{1B} \to e_\star^{1B} := E^{1B}(p_\star^B)$. The ODE (20) for e_t^{0B} can be written as $\dot{e}_t^{0B} = F_t^{0B}(e_t^{0B})$, where

$$F_t^{0B}(x) := \tfrac{1}{2}x^2 - \tfrac{1}{2}(e_t^{1B})^2 + (r + e_t^{1A})x$$

are continuously differentiable functions converging uniformly to the function $F_\star^{0B}(x) = \frac{1}{2}x^2 - \frac{1}{2}(e_\star^{1B})^2 + (r + e_\star^{1A})x$ as $t \to \infty$. Since $F_\star^{0B}$ is a quadratic polynomial with a positive leading coefficient, negative intercept, and $F_\star^{0B}(e_\star^{1B}) > 0$, it has a unique positive root $e_\star^{0B} \in (0, e_\star^{1B})$, and $(F_\star^{0B})'(e_\star^{0B}) > 0$. Applying Lemma 23, we conclude that $e_t^{0B} \to e_\star^{0B}$.

Since $e_\star^{0B} < e_\star^{1B}$, it must be that $p_\star^B \leq 1$, because otherwise $\dot{p}_t^B$ would necessarily be negative for t large (which would prevent it from approaching to 1). We conclude that $(e_\star^{1A}, e_\star^{1B}, e_\star^{0B}, p_\star^B)$ is a critical point of the ODE (20) with $p_\star^B < 1$, and thus, by Lemma 28, $(e_\star^{1A}, e_\star^{1B}, e_\star^{0B}, p_\star^B) = (e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$. □

Proof of Lemma 30. In any Markov perfect Bayesian equilibrium, p_t^B must be monotonous as otherwise there would be $0 < t_1 < t_2$ such that $p_{t_1}^B = p_{t_2}^B$, but $\dot{p}_{t_1}^B \neq \dot{p}_{t_2}^B$, which is not consistent with the Markov property.[23]

Since p_t^B is monotonous on a bounded range, it has to converge. The rest follows from the Lemma 34. □

D.4 Effort Over Time and Existence of Unique Equilibrium

Proposition 7 follows from the following lemmas:

Lemma 35. *Both efforts e_t^{1A} and e_t^{1B} increase over time, in fact $\dot{e}_t^{1A} > 0$ and $\dot{e}_t^{1B} > 0$ for all $t \geq 0$.*

[23] Indeed, p_t^B is the only state in the game, so if p_t^B is the same at the two times, then also e_t^{1A}, e_t^{1B}, and e_t^{0B} has to be the same. But that implies that also $\dot{p}_t^B$ is the same at the two times.

Proof. Define $T = \inf\{t \geq 0 : \dot{e}_t^{1A} > 0 \text{ and } \dot{e}_t^{1B} > 0\}$. Since the direction in which the solution has to converge to the steady state satisfies $\frac{\nu^{e1A}}{\nu^{pB}} > 0$ and $\frac{\nu^{e1B}}{\nu^{pB}} > 0$, both of the efforts are increasing for t large, and so T is finite. For contradiction suppose that at least one of the inequalities $\dot{e}_t^{1A} > 0$ and $\dot{e}_t^{1B} > 0$ is violated at some time $t \geq 0$. Then $\dot{e}_T^{1A} \geq 0$ and $\dot{e}_T^{1B} \geq 0$, and at least of one of the inequalities is binding:

- Either $\dot{e}_T^{1A} = 0$, and then taking the derivative of the formula for $\dot{e}_t^{1A}$ we obtain

$$\ddot{e}_T^{1A} = -(\dot{p}_T^B e_T^{1B} + p_T^B \dot{e}_T^{1B})(1 - e_T^{1A}) < 0,$$

  which is however a contradiction with the fact that $\dot{e}_t^{1A} > 0$ for all $t > T$.

- Or $\dot{e}_T^{1A} > 0$ and $\dot{e}_t^{1B} = 0$, in which case taking the derivative of the formula for $\dot{e}_t^{1B}$ we obtain

$$\ddot{e}_T^{1B} = -\dot{e}_T^{1A}(1 - e_T^{1B}) < 0,$$

  which is a contradiction with the fact that $\dot{e}_t^{1B} > 0$ for all $t > T$.

□

Lemma 36. *The effort e_t^{0B} decreases over time, in fact $\dot{e}_t^{0B} < 0$ for all $t \geq 0$.*

Lemma 37. *The effort of player B is higher when he is successful, i.e. $e_t^{1B} > e_t^{0B}$ for any $t \in [0, +\infty)$.*

The proof of these lemmas is analogous to the proof of corresponding properties in the symmetric case. Likewise the proof of the Proposition .

D.5 Comparison of Efforts

Proof of Proposition 8. First of all, the inequality $e_t^{1A} < e_t^{1B}$ holds at the steady-state. Indeed, the steady-state efforts are given by the following equations

$$e_*^{1A} = f^{-1}(p_*^B e_*^{1B})$$
$$e_*^{1B} = f^{-1}(e_*^{1A}),$$

where $f(x) = \frac{x^2}{2(1-x)}$ is the function analyzed in Lemma 11. For contradiction suppose $e_*^{1B} \leq e_*^{1A}$. Then $p_*^B e_*^{1B} < e_*^{1A}$, and since the function f^{-1} is strictly increasing, the inequality is preserved by applying the function f^{-1} to it, so $e_*^{1A} < e_*^{1B}$. Contradiction.

Next, for contradiction suppose that $e_t^{1B} > e_t^{1A}$ was not true for all t; so far we only know that is has to be true for t large. Let $T \geq 0$ then be the smallest real number such that $e_t^{1B} > e_t^{1A}$ holds for all $t > T$. Then necessarily $e_T^{1B} = e_T^{1A} =: e$ and $\dot{e}_T^{1B} \geq \dot{e}_T^{1A}$. However,

$$\dot{e}_T^{1A} = \tfrac{1}{2}(e)^2 - (r + p_T^B e)(1 - e) > \tfrac{1}{2}(e)^2 - (r + e)(1 - e) = \dot{e}_T^{1B}.$$

Contradiction. □

Proof of Proposition 9. The continuation values $v_t^{1A,(B)}$ and v_t^{1B} are given by the following ODEs

$$\begin{aligned} -\dot{v}_t^{1A,(B)} &= \tfrac{1}{2}\left(e_t^{1A}\right)^2 - (r + e_t^{1B})v_t^{1A,(B)} \\ -\dot{v}_t^{1B} &= \tfrac{1}{2}(e_t^{1B})^2 - (r + e_t^{1A})v_t^{1B}. \end{aligned}$$

Again, we first look at the steady-state. We have

$$v_*^{1A,(B)} = \frac{\frac{1}{2}\left(e_*^{1A}\right)^2}{r + e_*^{1B}} < \frac{\frac{1}{2}(e_*^{1B})^2}{r + e_*^{1A}} = v_*^{1B},$$

where we used the inequality $e_*^{1A} < e_*^{1B}$ from Proposition 8. Thus, the inequality $v_t^{1A,(B)} < v_t^{1B}$ necessarily holds for t large. Suppose that the inequality does not hold for all $t \geq 0$. Let $T \geq 0$ be the smallest real number such that $v_t^{1A,(B)} < v_t^{1B}$ holds for all $t > T$. Then necessarily $v_T^{1A,(B)} = v_T^{1B} =: v$ and $\dot{v}_t^{1A,(1)} \leq \dot{v}_t^{1B}$. However,

$$\dot{v}_T^{1A,(1)} = -\tfrac{1}{2}\left(e_T^{1A}\right)^2 + (r + e_T^{1B})v > -\tfrac{1}{2}(e_T^{1B})^2 + (r + e_T^{1A})v = \dot{v}_t^{1B},$$

where we used the inequality $e_T^{1A} < e_T^{1B}$ from Proposition 8. Contradiction. □

E Proofs – Patent Race with Optional Revelation

E.1 Never Reveal Second

Proof of Proposition 10. Suppose one player has revealed success already. Without loss of generality, let it be player A. First, the strategy to never reveal is an equilibrium strategy of player B. Indeed, player B's continuation value implied by the strategy to never reveal is $v^{1B}(1, p_t)$, while his continuation value of revealing is $v^{11} = v^{1B}(1, 1)$. Applying Lemma 33, we conclude that $e^{1B}(1, p_t) < E^{1B}(\sup_{s \geq t} p_s) < E^{1B}(1) = e^{1B}(1, 1)$, and so $v^{1B}(1, p_t) > v^{1B}(1, 1)$. Thus, a player has indeed no incentive to reveal.

To show that not revealing second is the only equilibrium, we need to consider any strategy of player B over revealing second, because player B's strategy over revealing impacts his rival's effort, and so it impacts his own incentive to reveal. The efforts and continuation values of the two players follow the same differential equations as those in a private information game with one player being known to be successful, except that the dynamic of p_t^B is influenced by player B's strategy over revelation. If player B is expected to reveal with a positive probability once being successful, then his rival's posterior belief p_t^B grows slower (or even falls), than it would in the game without revelation; and in the event of player B revealing, it jumps to 1 and remains there. We do not need to describe the exact process of p_t; what is relevant is that p_t^B is less than 1 with a positive probability for a while. We can follow the reasoning from the proof of Lemma 33 and generalize its results for a stochastic process p_t^B, and obtain the estimate that player B's continuation value while being in state 1 is strictly more than $1 - E^{1B}(1)$, which is the continuation value he would get after revealing.

In conclusion, regardless of what player B's strategy over revealing as second is, he has an incentive not to reveal. Thus, the only equilibrium strategy player B can have is not to reveal second. □

E.2 No-revelation Equilibrium

Proof of Proposition 11. First, the condition $v^{1A}(1,p) \leq v^1(p)$ for all $p \in [0, p_*)$ is necessary for a no-revelation equilibrium to exist. Indeed, suppose that a no-revelation equilibrium exists and that $v^{1A}(1,p) > v^1(p)$ for some $p \in [0, p_*)$. Then player A has an incentive to reveal the arrival of a breakthrough at time t such that $p_t^A = p$. Contradiction.

Assume that $v^{1A}(1,p) \leq v^1(1,p)$ for all $p \in [0, p_*)$, and suppose that both players have the strategy to never reveal. We check that none of the players has an incentive to deviate. Given that players do not reveal, their efforts and continuation values are identical to those from the private information version of the game (without revelation). In particular, the continuation value of a successful player at time t is $v^1(p_t)$. In contrast, if a successful player deviated and revealed, his continuation value would be $v^{1A}(1, p_t)$, which is no more than $v^1(p_t)$ by the assumption.

The no-revelation equilibrium is unique, as the effort levels have to correspond to the unique solution of the private information version of the game (Proposition 3).

□

E.3 Instant-revelation Equilibrium

Proof of Proposition 12. Suppose that both players have the strategy to reveal the arrival of a breakthrough instantly (unless the rival has revealed already), and none has done so by time $t \geq 0$. We will prove that either player has no incentive to deviate if and only if inequality (4) is satisfied. Until either of the players reveals, the game is static in the sense that each player is certain that his rival is unsuccessful ($p_t = 0$).

The effort $e_\bullet^0$ of an unsuccessful player (and the corresponding continuation value $v_\bullet^0 = v^{1A}(1,0) - e_\bullet^0$) is characterized by the equation (5). The RHS of equation (5) is a quadratic polynomial of $e_\bullet^0$, which is negative at $e_\bullet^0 = 0$ and positive at $e_\bullet^0 = v^{1A}(1,0)$. As a result, it has a unique root $e_\bullet^0$ in the interval $(0, v_\bullet^{1A}(1,0))$.

If player $j \in \{A, B\}$ deviates and does not reveal, then his continuation value is $\tilde{v}^1$ given by the equation

$$0 = \tfrac{\alpha}{2}\left(v - \tilde{v}^1\right)^2 + e_\bullet^0 v^{1B}(1,0) - (r + e_\bullet^0)\tilde{v}^1, \tag{24}$$

whose RHS is a quadratic polynomial of $\tilde{v}^1$, which is positive at $\tilde{v}^1 = 0$ and negative at $\tilde{v}^1 = v$. Thus, there is unique $\tilde{v}^1 \in (0, v)$ that solves the equation (24). It also follows that $\tilde{v}^1 \leq v^{1A}(1,0)$ if and only if the RHS of (24) evaluated at $\tilde{v}^1 = v^{1A}(1,0)$ is nonpositive. That gives us the inequality (4). □

E.4 Mixed-strategy Equlibria

Proof of Lemma 3. Consider a symmetric equilibrium, and let us analyze the situation from the perspective of of player A. If $p_t = 0$, then the claim is trivial as there is nothing to be revealed. In the rest of the proof, consider $p_t > 0$. Let us distinguish cases based on how the continuation value $v^{1A}(1, p_t)$ that player A (when being successful) obtains by revealing compares with the continuation value $v^{1A}_\bullet(p_t)$ that he obtains by not revealing:

- Case $v^{1A}(1, p_t) < v^{1A}_\bullet(p_t)$: Then it does not pay off to reveal, and necessarily $\theta_\bullet(p_t) = 0$.
- Case $v^{1A}(1, p_t) > v^{1A}_\bullet(p_t)$: Then player A, if successful, would reveal already before time t, and p_t would have to be 0.
- Case $v^{1A}(1, p_t) = v^{1A}_\bullet(p_t)$: For contradiction, suppose that $\theta_{\bullet t} = +\infty$. Hence, the chance with which player B reveals in the time interval $[t, t+\Delta t]$ is an arbitrarily large multiple of Δt as Δt goes to 0. If player A reveals, he gets the continuation value $v^{1A}(1, p_t)$, whilst if he waits an arbitrarily short time Δt, he likely ends up with the continuation value $v^{1B}(1, p_t) > v^{1A}(1, p_t)$ (the inequality follows from Proposition 8). Thus, player A prefers to postpone revelation in that case, implying $\theta_\bullet(p_t) = 0$.

□

Proof of Lemma 4. Recall that p^j_t is the posterior probability of player j being in state 1 at time t. Unlike in the case of the game without the option to reveal, the posterior here is conditioned not only on the fact that player j has not patented, but also on the fact that he has not revealed by time t, which I denote as event N^j_t.[24] Accordingly,

$$\begin{aligned}
p^j_{t+\Delta t} &= \mathbf{P}[x^j_{t+\Delta t} = 1 \mid x^j_{t+\Delta t} < 2, N^j_{t+\Delta t}] \\
&= \frac{\mathbf{P}[x^j_{t+\Delta} = 1, N^j_{t+\Delta t} \mid x^j_t < 2, N^j_t]}{\mathbf{P}[x^j_{t+\Delta t} < 2, N^j_{t+\Delta t} \mid x^j_t < 2, N^j_t]} \\
&= \frac{(1-p^j_t)e^{0j}_{\bullet t}\Delta t + p^j_t(1 - e^{1j}_{\bullet t}\Delta t) - \theta^j_{\bullet t}\Delta t}{1 - p^j_t e^{1j}_{\bullet t}\Delta t - \theta^j_{\bullet t}\Delta t} + o(\Delta t).
\end{aligned}$$

Taking the derivative with respect to Δt and evaluating at $\Delta t = 0$, we conclude

$$\begin{aligned}
\dot{p}^j_t &= \left((1-p^j_t)e^{0j}_{\bullet t} - p^j_t e^{1j}_{\bullet t} - \theta^j_{\bullet t}\right) \cdot 1 + p^j_t \cdot \left(p^j_t e^{1j}_{\bullet t} + \theta^j_{\bullet t}\right) \\
&= (1-p^j_t)(e^{0j}_{\bullet t} - p^j_t e^{1j}_{\bullet t} - \theta^j_{\bullet t}).
\end{aligned}$$

□

Proof of Lemma 5. Lemma 3, $\theta_\bullet(p_t)$ has to be finite. Hence, it follows that p_t is a continuous function of time (it cannot drop discretely). However, then p_t can never be decreasing, as otherwise there would be times $t_1 < t_2$ such that $p_{t_1} = p_{t_2}$, but $\dot{p}_t = e^0_\bullet(p_t) - \theta_\bullet(p_t) - p_t e^1_\bullet(p_t)$

[24] In fact, the probability is conditioned on the fact that neither of the players has patented or revealed, but it has no impact on the calculation.

is positive at t_1 and negative at t_2, which is impossible.[25] As a result, $\dot{p}_t \geq 0$ and so $\theta_\bullet(p_t) \leq e^0_\bullet(p_t) - p_t e^1_\bullet(p_t)$. □

Proof of Lemma 6. If a player has the strategy to reveal making a breakthrough with certainty at time $t = 0$ ($\theta_{\bullet 0} = e^0_{\bullet t}$), then p_t remains constantly at zero. Indeed, the Markov property then implies that until one of the players reveals, the players have to choose the same action at all times because the payoff relevant state p_t does not change. So if the player reveals with certainty at time $t = 0$, then the equilibrium has to be the *instant-revelation equilibrium.*

Conversely, if the player does not have the strategy to reveal with certainty at time $t = 0$, then $\dot{p}_0 > 0$, and since p_t is nondecreasing (Lemma 5), it follows that $p_t > 0$ for all $t > 0$, and thus he does not reveal with certainty, as by Lemma 5, $\theta_{\bullet t} \leq e^0_{\bullet t} - p_t e^1_{\bullet t} < e^0_{\bullet t}$. □

Proof of Lemma 7. Consider any symmetric equilibrium other than the instant-revelation equilibrium, and suppose that none of the players has revealed by time t. First of all, a player always has the option to reveal, and so $v^1_{\bullet t} \geq v^{1A}(1, p_t)$. If this inequality is strict, the player will not reveal, and so $\theta_{\bullet t} = 0$.

At every moment, a player chooses between revealing and not revealing. However, a player can never do strictly better by revealing, because in that case he would have to reveal with certainty, and that is not possible in equilibrium according to Lemma 6.

The continuation value of a successful player (before anyone has revealed) $v^1_{\bullet t}$ is given by the following recursive formula:[26]

$$v^1_{\bullet t} = \max_{e \geq 0} \left\{ v\, e\Delta t - \tfrac{\alpha}{2}(e)^2 \Delta t + v^{1B}(1, p_t)\theta_{\bullet t}\Delta t + [1 - (r + e + \theta_{\bullet t} + p_t e^1_{\bullet t})\Delta t] v^1_{t+\Delta t} + o(\Delta t) \right\}.$$

The first order condition yields $e = e^1_{\bullet t} = \frac{1}{\alpha}(v - v^1_{\bullet t})$. Plugging it back,

$$v^1_{\bullet t} = v^1_{\bullet\, t+\Delta t} + \left[\tfrac{\alpha}{2}(e^1_{\bullet t})^2 + v^{1B}(1, p_t)\theta_{\bullet t} - (r + \theta_{\bullet t} + p_t e^1_{\bullet t}) v^1_{\bullet t} \right] \Delta t + o(\Delta t),$$

and after subtracting $v^1_{\bullet\, t+\Delta t}$, dividing by Δt, and letting $\Delta t > 0$ to zero, we obtain the equation (6).

The derivation of the equation (7) for $\dot{v}^0_{\bullet t}$ is analogous, and the equation (8) for $\dot{p}_t$ follows from Lemma 4. □

Proof of Corollary 2. By Lemma 7, $\theta_{\bullet t} > 0$ implies $v^1_{\bullet t} = v^{1A}(1, p_t)$. As the function $t \mapsto \theta_{\bullet t}$ is by definition a right-continuous, $\theta_{\bullet t+\Delta t} > 0$ for all $\Delta t \geq 0$ small enough, and thus also $v^1_{\bullet\, t+\Delta t} = v^{1A}(1, p_{t+\Delta t})$ for all $\Delta t \geq 0$ small enough. Thus, $\dot{v}^1_{\bullet t} = v^{1A}_{\partial p}(1, p_t)\dot{p}_t$. Substituting this into the equation (6), we obtain the result. □

Proof of Lemma 8. Given that players stop revealing at time T, the continuation game is identical to the game without the option to reveal. Accordingly, the continuation value of

[25] This proof is based on the restriction to Markov Perfect equilibria.

[26] Note that this formula is evaluated as if the player whose value function is being calculated did not reveal, because if he reveals, then he is indifferent from revealing, and so it has no impact on the utility.

player A at time T is $v^1(p_T)$. Clearly, $v^1(p_T) \geq v^{1A}(1, p_T)$, because otherwise player A would be tempted to reveal at time T. At the same time, $v^1(p_T) \leq v^{1A}(1, p_T)$, because otherwise player A would have an incentive not to reveal at some time $t < T$. (This follows from the continuity of the functions $v^{1A}(1, p_t)$ and $v^1_\bullet(p_t)$.) Consequently, $v^{1A}(1, p_T) = v^1(p_T)$. □

Proof of Proposition 13. Consider a mixed-revelation equilibrium, let $T > 0$ be the time at which players stop revealing at all. By Lemma 8, $v^{1A}(1, p_T) = v^1(p_T)$. The rest follows from Lemma 7 and Corollary 2. The equation (10) is obtained from equation (9),

$$- v^{1A}_{\partial p}(1, p_t)(1 - p_t)(e^0_{\bullet t} - p_t e^{1A}(1, p_t) - \theta_{\bullet t}) \\ = \tfrac{\alpha}{2}(e^{1A}(1, p_t))^2 + \theta_{\bullet t} v^{1B}(1, p_t) - (r + \theta_{\bullet t} + p_t e^{1A}(1, p_t))v^{1A}(1, p_t),$$

$$0 = \tfrac{\alpha}{2}(e^{1A}(1, p_t))^2 + \theta_{\bullet t} v^{1B}(1, p_t) - (r + e^0_{\bullet t})v^{1A}(1, p_t) + (e^0_{\bullet t} - p_t e^{1A}(1, p_t) - \theta_{\bullet t})\tilde{v}^{1A}_t$$

Solving for $\theta_{\bullet t}$ concludes the proof. □

Proof of Lema 9 (partially numerical). We first show that a mixed-revelation equilibrium cannot coexists with any of the pure-strategy equilibria.

Suppose that a no-revelation equilibrium exists. Then by Lemma 2, $v^{1A}(1, p) < v^1(p)$ for all $p \in (0, 1)$, and so there does not exist $\bar{p} \in (0, 1)$ at which $v^{1A}(1, \bar{p}) = v^1(\bar{p})$, and so by Lemma 8, a mixed-revelation equilibrium cannot exist.

Next, we consider the case in which an instant-revelation equilibrium exists. It can be shown numerically that, in that case, there exists $\underline{p} \in [0, \overline{p})$ at which the system of ODEs (11)-(12) has a steady-state (Figure 7). When going backwards in time from $p_T = \overline{p}$, p_t will never drop below $\underline{p}$, and thus no mixed-revelation equilibrium exists.

On the other hand, if none of the pure-stretegy equilibria exists, then the unique mixed-revealing equilibrium is characterized by Proposition 13. □

Proof of Lemma 10 (partially numerical). First, we show that players must stop revealing definitively at some $T > 0$. For contradiction, suppose that players never stop revealing definitively. Then there exists p to which p_t converges and $\theta_\bullet(p) > 0$. This p has to correspond to a steady-state of the system of ODEs (11)-(12). However, analyzing the ODE, it can be shown that this steady-state can be classified as a source, which means that no solution of the ODE can converge towards it. The situation is illustrated in Figure 15.

Let T be the time at which players stop revealing definitively. Consider the longest interval $[t_0, T)$ on which the players mix over revelation (since $t \mapsto \theta_\bullet$ is right-continuous, the longest interval exists).[27] Then the equation (10) holds at $t = t_0$ and $\theta_{\bullet t_0} > 0$. However, then for $\Delta t > 0$ small enough, a player has an incentive to reveal at time $t = t_0 - \Delta t$ given that $\theta_{\bullet t} = 0$. It must be the case that $t_0 = 0$. □

Proof of Proposition 14 (partially numerical). By Lemma 10 the mixed-revelation equilibrium is the unique candidate for equilibrium involving mixed strategies. By Lemma 9 the mixed-revelation equilibrium exists if and only if any of the pure-strategy equilibria does not.

[27] In fact, we also use the assumption that $p \mapsto \theta_\bullet(p)$ is piecewise continuous.

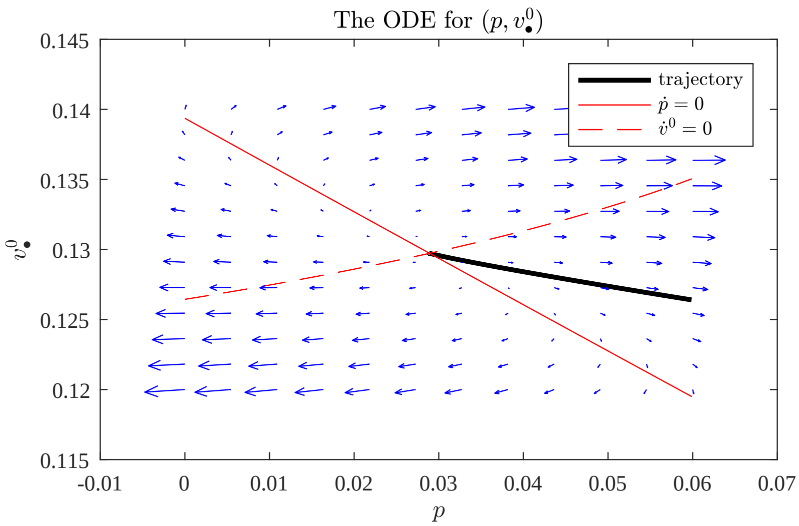

Figure 15: The neighborhood of the steady-state of the system of ODEs (11)-(12) for $r' = 0.2$. The heavy solid line shows the trajectory of the solution ending in the point $(\overline{p}, v^0(\overline{p}))$; when going back in time, the trajectory converges towards the steady-state. The derivative $\dot{p}$ is positive above the solid line, and $\dot{v}^0_\bullet$ is positive above the dashed line. The two lines cross if and only if an instant-revelation equilibrium exists ($r' > 0.1707$).

As a result, there always exists a unique equilibrium. Its type depends on the parameters. The thresholds are found numerically. □

Abstrakt

Tento článek vyšetřuje roli soukromé informace v soutěži v patentování. Běžný předpoklad v literatuře týkající se soutěže v patentování, že firmy při soutěžení znají pokrok svého soupeře, je diskutabilní, protože se výzkum často děje za zavřenými dveřmi. Soukromá informace zásadně mění dynamiku soutěže a vede k otázce, zda je v zájmu firem dobrovolně zveřejňovat svůj pokrok za účelem odrazení soupeře. Analýza pobídky naznačuje, že v zájmu firmy je zveřejnit pokrok pouze pokud ji dává dostatečný náskok oproti soupeři.

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